(Robust) Stability of Time-Delay Systems: Recent Results and Open Questions

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Dynamical systems: models for the evolving world...

- Car trailer system
  - Video: `https://www.youtube.com/watch?v=4jk9H5AB41M`
- Aircraft
  - Video: `https://www.youtube.com/watch?v=4UfmsqtTGa0`
- Car active suspension system
  - Video: `https://www.youtube.com/watch?v=kRt7H0k8A4k`
- Building
  - Video: `https://www.youtube.com/shorts/rJ72LruGgyU`

A general form of a dynamic system can be given as follows:

$$\dot{x}(t) = \frac{dx}{dt} = f(t, x(t), u(t); \beta), \quad t \geq t_0$$

where

- $x(t) \in \mathbb{R}^n$ is the state vector,
- $u(t) \in \mathbb{R}^m$ is the system input and
- $\beta \in \mathbb{R}^p$ is the vector of system parameters.

The state vector consists of minimal number of set of values that you need to describe your system and it is a unique minimal description of the system you care about.

- $x_{\text{ref}} = \text{const}$
- Find $u(t) = \gamma(t, x(t))$ such that the closed-loop system (CLS) $\dot{x}(t) = f_p(t, x(t), \gamma(t, x(t))) = \tilde{f}(t, x(t))$ has a desired behavior.
- Desired CLS behavior:

$$\left.\begin{array}{ll} - & x_{\text{ref}} \text{ an equilibrium point, i.e. } \tilde{f}(t, x_{\text{ref}}) = 0 \\ - & \text{Convergence: } \lim_{t\to\infty} x(t) = x_{\text{ref}} \\ - & \text{Start close } \implies \text{ stay close} \end{array}\right\} \equiv \begin{array}{c}\text{Asymptotic}\\ \text{stability}\end{array}$$

Asymptotic Stabilization Problem

Find $u(t) = \gamma(t, x(t))$ such that x_{ref} is an asymptotically stable equilibrium point of $\dot{x}(t) = f_p(t, x(t), \gamma(t, x(t))) = \tilde{f}(t, x(t))$.

- Coordinate transformation:

$$\begin{aligned} e(t) =& x(t) - x_{\text{ref}} \; (x = x_{\text{ref}} \iff e = 0), \\ \dot{e}(t) =& \dot{x}(t) - \dot{x}_{\text{ref}} = \dot{x}(t) = \tilde{f}(t, e(t) + x_{\text{ref}}) = f(t, e(t)) \end{aligned}$$

$\rightarrow$ x_{ref} goes as a parameter in the function

Asymptotic Stabilization Problem

Find $u(t) = \gamma(t, e(t))$, $e(t) = x(t) - x_{ref}$ such that $e = 0$ is an asymptotically stable equilibrium point of $\dot{e}(t) = f(t, e(t))$.

- $x_{\text{ref}}(t)$
- Desired CLS behavior:
  - on trajectory $\Longrightarrow$ stay on trajectory
  - convergence to trajectory
  - start close $\Longrightarrow$ stay close
- Coordinate transformation:

$$\begin{aligned} e(t) &= x(t) - x_{\text{ref}}(t), \\ \dot{e}(t) &= \dot{x}(t) - \dot{x}_{\text{ref}}(t) = \underbrace{f_p(t, e(t) + x_{\text{ref}}(t), u(t)) - \dot{x}_{\text{ref}}(t)}_{:=\bar{f}_p(t,e(t),u(t))} \end{aligned}$$

Asymptotic Stabilization Problem

Find $u(t) = \gamma(t, e(t))$, $e(t) = x(t) - x_{ref}(t)$ such that $e = 0$ is an asymptotically stable equilibrium point of $\dot{e}(t) = \bar{f}_p(t, e(t), \gamma(t, e(t))) = f(t, e(t))$.

Asymptotic Stabilization Problem (Regulation Problem)

Find $u(t) = \gamma(t, e(t))$, $e(t) = x(t) - x_{ref}$ such that $e = 0$ is an asymptotically stable equilibrium point of $\dot{e}(t) = f(t, e(t))$.

Asymptotic Stabilization Problem (Tracking Problem)

Find $u(t) = \gamma(t, e(t))$, $e(t) = x(t) - x_{ref}(t)$ such that $e = 0$ is an asymptotically stable equilibrium point of $\dot{e}(t) = f(t, e(t))$.

- Therefore, from now on, for simplicity, we will
  - either consider $\dot{x} = f(t, x)$, $x = 0$ equilibrium point
  - or consider $\dot{x} = f(x)$, $x = 0$ equilibrium point.

$$\dot{x} = f(x), \; f \; : \; D \subset \mathbb{R}^n \to \mathbb{R}^n$$

Definition (Stability)

$x = 0$ is called a **stable** equilibrium point if and only if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \; \forall \geq 0.$$

In this stability definition, it shall always be possible to keep the system state arbitrarily close to the equilibrium point by starting sufficiently close.

Definition (Asymptotic Stability)

The equilibrium point $x = 0$ is (locally) asymptotically stable if and only if

i) $x = 0$ is stable,

ii) $\exists r > 0$ such that $\|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0$ (convergence).

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Definition (Region of Attraction)

The region of attraction R_a (also called region of asymptotic stability, domain of attraction or basin) is the set of all points in $D \subset \mathbb{R}^n$ such that the solution the solution of $\dot{x} = f(x)$, $x(0) = x_0$ is defined for all $t \geq 0$ and converges to the equilibrium point $x = 0$ as $t \to \infty$.

According to this defn., if $R_a = \mathbb{R}^n$, then $x = 0$ is globally asymptotically stable.

Definition (Global Asymptotic Stability)

The equilibrium pt. $x = 0$ is globally asymptotically stable (GAS) if and only if

i) $x = 0$ is stable,

ii) $\forall x(0) \in \mathbb{R}^n$ we have $\lim_{t\to\infty} x(t) = 0$ (global convergence)

GAS guarantees that $x = 0$ is the only equilibrium point!

Definition (Global Asymptotic Stability)

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i) $x = 0$ is stable,

ii) $\forall x(0) \in \mathbb{R}^n$ we have $\lim_{t\to\infty} x(t) = 0$ (global convergence)

Question: Why there is a need of a requirement of “stability" in the GAS definition? Is convergence not sufficient to imply stability?

The answer of this question is “yes", for linear systems; but “no", for nonlinear systems. So, Convergence $\not\Rightarrow$ Stability in general. Let us consider the Vinograd's counter example.

Example (Vinograd's Counter-Example)

$$\dot{x} = \frac{x^2(y-x)+y^5}{r^2(1+r^4)}, \quad \dot{y} = \frac{y^2(y-2x)}{r^2(1+r^4)} \quad \textit{where} \quad r^2 = x^2+y^2$$

In this system, the equilibrium point is convergent but it is **not** stable! See the next slide!

Stability Definitions and Solution Characteristics

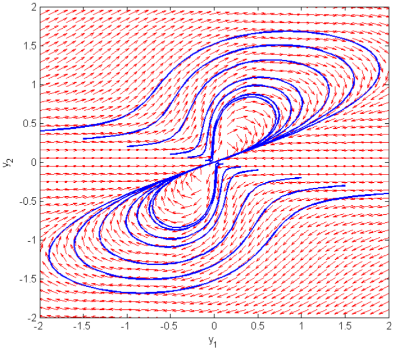


Figure: Vector Field of Vinograd's System.

Figure Credit (See Page 287):

A. Mironchenko, *Input-to-State Stability: Foundations and Applications* Springer Nature, 2023.

See also KYP Lectures (L.3.3-16:07): `https://youtu.be/B5PgJgdlz_Y?list=PLdeo5-jZaFjP9HDqhSt3wzaaVPpRydA9Y&t=967`

A particular type of stability exists differing in the way they dissipate along solutions.

Definition (Exponential Stability)

The equilibrium point $x = 0$ is (locally) exponentially stable if and only if $\exists r, k, \lambda > 0$ such that

$$\|x(0)\| < r \Rightarrow \|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall t \geq 0.$$

Again, if $R_A = \mathbb{R}^n$, then $x = 0$ is globally exponentially stable and, again, this also implies that $x = 0$ is the only equilibrium point.

Definition (Globally Exponential Stability)

The equilibrium point $x = 0$ is globally exponentially stable (GES) if and only if $\exists k, \lambda > 0$ such that, $\forall x(0)$, we have

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall t \geq 0.$$

Note that, there is no explicit assumption of stability and convergence because, the estimate guarantee these properties!

Stability Definitions and Solution Characteristics

Note that, the following implications hold:

- Exponential Convergence $\Rightarrow$ Stability. Because, $\forall\varepsilon > 0$, we can always find $\delta = \frac{\varepsilon}{k}$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| \leq k\|x(0)\|\underbrace{e^{-\lambda t}}_{\leq 1} < k\frac{\varepsilon}{k} = \varepsilon$$

- Exponential Convergence $\Rightarrow$ Asymptotic Stability. Because

$$\lim_{t\to\infty}\|x(t)\| \leq \lim_{t\to\infty} k\|x(0)\|e^{-\lambda t} = k\|x(0)\|\lim_{t\to\infty} e^{-\lambda t} = 0$$

  holds for all $\|x(0)\| < \infty$ which implies convergence.

However, the converse of the last implication does not hold in general:

- Asymptotic Stability $\not\Rightarrow$ Exponential Convergence. For example, the system

$$\dot{x} = -x^2, \; x(0) = 1$$

  has the solution $x(t) = \frac{1}{1+t}$ which is AS but not ES.

System Energy and "Energy-Like" Functions

Lyapunov's indirect method analyze the (local) stability of the equilibrium point of the systems. By Lyapunov's direct method, we are able to analyze the stability properties of an equilibrium point which enables us to ensure stability, asymptotic stability and exponential stability of these equilibrium points in "local" and "global" sense.

The motivation for Lyapunov's direct method comes from the consumption of energy of the system. For example, Hamiltonian systems governed by Hamilton's equations.

Consider a two dimensional system is a scalar function of the state and we will denote this by V.

- The energy of the equilibrium point is zero.
- Now, let us draw a curve to all points in the state space where the energy has the same constant value, say c_1. Let us, then draw another curve through all the points in the state space where the energy has another constant where c_2 which is greater than c_1. These curves are called **level curves** or **level surfaces** (in $\mathbb{R}^n$). These curves/surfaces represent constant energy levels for the system.

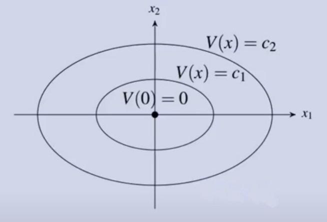


Figure: The Level Surfaces Representing Constant Energy Levels $V(x) = c_i$, $i = 1, 2$ $(0 < c_1 < c_2)$.

Level surfaces $V(x) = c_i$, $0 < c_1 < c_2 < c_3 < \ldots$ are the surfaces that represent constant energy levels!

Question: How to choose this energy function to analyze the stability properties of the equilibrium point at the origin?

Recall that $x = 0$ is an equilibrium point of $\dot{x} = f(x)$. What we do is that we study the time evolution of the energy of the system. Specifically, we study the energy evolving along the system trajectories.

System Energy and “Energy-Like" Functions

If the system trajectory moves towards level curves representing higher energy levels, then this corresponds to moving further and further away from the equilibrium point which should suggest that the origin is unstable as shown in the following figure.

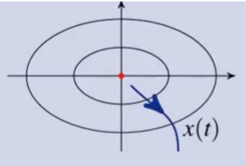


Figure: Energy increases along $x(t)$.

In this case, the time evolution of the energy will be positive:

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) > 0$$

Remember that, we compute the time evolution of the energy along the system trajectories by simply taking the time derivative of the energy function V:

$$\dot{V}(x) = \frac{dV}{dt} = \left\langle \frac{\partial V}{\partial x}, \frac{dx}{dt} \right\rangle = \frac{\partial V}{\partial x} \cdot f(x)$$

Remember also that, this is also called the directional derivative of the function V along the vector field f (or Lie derivative).

System Energy and "Energy-Like" Functions

If the system trajectory moves along the level curves, then this means that the system trajectories intersect curves representing lower and lower energy levels until the energy becomes zero which is at the origin which corresponds to negative time evolution of energy:

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) < 0$$

This indicates that the origin is an asymptotically stable equilibrium point as shown in the following figure.

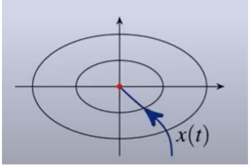


Figure: Energy decreases along $x(t)$.

Note that, we do not need to solve $\dot{x} = f(x)$ in order to see how the energy increases or decreases along $x(t)$. The sign of the directional derivative of the energy function V does this job.

If the time derivative of the energy function is allowed to be zero, then the energy is constant at some future time meaning that the system trajectory moves along one of the level curve. So when the time derivative of the energy function is either negative or 0, mathematically speaking:

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) \leq 0$$

then we do not necesserily have convergence but the behavior is similar to that of a stable equilibrium point as shown in the following figure.

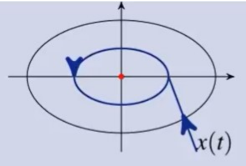


Figure: Energy decreases or is constant along $x(t)$.

These intuitive observations hold for mechanical and electrical systems, which have a well-defined energy concept.

Lyapunov formalized and generalized these intuitive observations for general dynamical systems in his doctoral thesis at University of Kharkiv:

A. M. Lyapunov, *The General Problem of the Stability of Motion (In Russian)*, University of Kharkiv Doctoral Dissertation, 1892.

The theorems presented here, which constitute Lyapunov's direct method, are valid for general systems, not only for electrical or mechanical systems for which we have a well-defined energy concept. Instead of the energy function, we must therefore use an “energy-like" function. This “energy-like" function has to serve as a generalized energy function such that it satisfies certain conditions and, therefore, we can use it to analyze stability of the general dynamical systems.

Consider the TI system $\dot{x} = f(x)$ where $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, $x = 0 \in D$ is an equilibrium point of the system. To use Lyapunov's direct method, we need to find a function that can be used instead of energy to determine the stability properties of the equilibrium point. This function is called a Lyapunov function candidate.

Definition (Lyapunov Function)

$V : D \to \mathbb{R}$ is a **Lyapunov function** for $x = 0$ if and only if

i) $V \in \mathcal{C}^1$

ii) $V(0) = 0$ and $V(x) > 0$ in $D \setminus \{0\}$ (positive definiteness in D)

iii) $\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) \leq 0$ in D (negative semi-definiteness in D)

If, moreover,

iv) $\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) < 0$ in D $\setminus \{0\}$,

then, V is a **strict Lyapunov function** for $x = 0$.

Remark

If (i) and (ii) are satisfied, for a $V : D \to \mathbb{R}$, then V is called a ***Lyapunov function candidate****. In addition to (i) and (ii), when (iii) or (iv) are satisfied, the function V will no longer a "candidate" and named as Lyapunov function or strict Lyapunov function, respectively. Note also that, (iii) and (iv) are the only conditions where the system dynamics come into the criterion. The first two conditions only work out the function itself and the system dynamics do not come in here.*

Now, we are ready to present the sufficient conditions for stability and asymptotic stability of the equilibrium point.

Theorem (Lyapunov's Direct Method)

- *If there exists a* ***Lyapunov function*** *for $x = 0$, then $x = 0$ is* ***stable****.*
- *If there exists* ***strict Lyapunov function*** *for $x = 0$, then $x = 0$ is* ***asymptotically stable****.*

Question: How to apply Lyapunov's direct method for general $\dot{x} = f(x)$?

1) Choose a Lyapunov function "candidate" $V(x)$
   - For electrical/mechanical systems

$$V(x) = \text{Total Energy}$$

   - For others quadratic forms are generally used:

$$V(x) = \frac{1}{2}x^T P x = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^\top \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{12} & p_{22} & \dots & p_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ p_{1n} & p_{2n} & \dots & p_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

$$= \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + \dots$$

   - There are also some other methods for choosing $V(x)$

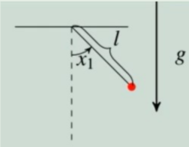
2) Determine whether $V(x)$ is a **Lyapunov function** or a **strict Lyapunov function** for the equilibrium point.
3) If the answer is "yes", then the equilibrium point is **stable** or **asymptotically stable**. If the answer is "no", then go back to step 1.

Remark

Failing to establish a Lyapunov function does not mean that the equilibrium point is unstable. The result of Lyapunov's direct method contains sufficent conditions not necessary conditions. There exists instability theorem for establishing an equilibrium point is unstable, e.g. Chetaev's theorem (see Theorem 4.3 in Khalil's book), but it will not be covered here.

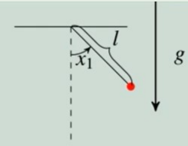
Example (Pendulum without Friction)

Now, we will consider the system governed by pendulum without friction:



$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Example (Pendulum without Friction)



$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin\, x_1$$

Our Lyapunov function "candidate" may be

$$\begin{aligned} V(x) &= V_{\text{pot}}(x) + V_{\text{kin}}(x) \\ &= -\int_0^{x_1} -\frac{g}{\ell}\sin\, y\, dy + \frac{1}{2}x_2^2 = \frac{g}{\ell}(1 - \cos\, x_1) + \frac{1}{2}x_2^2. \end{aligned}$$

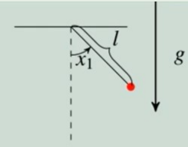
Note that, $V \in \mathcal{C}^1$. We choose

$$D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\} \subset \mathbb{R}^2,$$

and this implies

$$V(0) = 0 \text{ and } V(x) > 0,\ \forall x \in D \setminus \{0\}.$$

Example (Pendulum without Friction)



$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin\, x_1$$

Differentiating V along the solutions of the system yields to

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \frac{g}{\ell}\sin\, x_1 x_2 + x_2\left(-\frac{g}{\ell}\sin\, x_1\right) = 0, \quad \forall x \in D$$

This makes sense, since this is a conservative system. Therefore

$$V \;:\; D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\} \to \mathbb{R}$$

is a Lyapunov function for $x = 0$ which tells us that $x = 0$ is a stable equilibrium point.

Lyapunov Functions and Lyapunov's Stability Theorem

Example (Pendulum with Friction)

Consider, now, the system governed by pendulum with friction:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{\ell}\sin\, x_1 - \frac{k}{m}x_2.\end{aligned}$$

For simplicity, let us take $m = 1$.

Since a friction force is acting on this system, the system is no longer a conservative system. Friction is a dissipative force, which draws energy from the system. Let us again choose the same Lyapunov function “candidate", which we know that $V \in \mathcal{C}^1$, V is positive definite in $D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\}$. Now, let us check the derivative of V along the solutions of the system:

$$\dot{V}(x) = \frac{g}{\ell}\sin\, x_1 x_2 + x_2\left(-\frac{g}{\ell}\sin\, x_1 - kx_2\right) = -kx_2^2 \leq 0, \quad \forall x \in D$$

which implies that $x = 0$ is a stable equilibrium point. Moreover, we know that $x = 0$ is an asymptotically stable equilibrium point.

Example (Pendulum with Friction)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - k\,x_2.$$

Let us replace the term $(1/2)x_2^2$ by more general quadratic form $(1/2)x^\top Px$ for some 2×2 positive definite symmetric matrix P:

$$V(x) = \frac{1}{2}x^T Px + \frac{g}{\ell}(1 - \cos x_1) = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{g}{\ell}(1 - \cos x_1)$$
$$= \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + \frac{g}{\ell}(1 - \cos x_1)$$

For the matrix P to be positive definite, the elements of P must be satisfy

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

The directional derivative of V along the solutions of the system yields to

$$\dot{V}(x) = \left(p_{11}x_1 + p_{12}x_2 \frac{g}{\ell}\sin x_1\right)x_2 + (p_{12}x_1 + p_{22}x_2)\left(-\frac{g}{\ell}\sin x_1 - kx_2\right)$$
$$= \frac{g}{\ell}(1 - p_{22})x_2 \sin x_1 - \frac{g}{\ell}p_{12}x_1 \sin x_1 + (p_{11} - p_{12}k)x_1x_2 + (p_{12} - p_{22}k)x_2^2$$

Example (Pendulum with Friction)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin\, x_1 - k\, x_2.$$

Now, we want to choose p_{11}, p_{12} and p_{22} such that $\dot{V} < 0$. Since the cross-product terms $x_2 \sin\, x_1$ and $x_1 x_2$ are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = kp_{12}$. With these choices, we have

$$p_{11}p_{22} - p_{12}^2 = p_{12}(k - p_{12}) > 0 \Rightarrow 0 < p_{12} < k \quad (\text{for } k > 0)$$

for $V(x) > 0$. Let us take $p_{12} = \frac{k}{2}$, then $\dot{V}(x)$ will be

$$\dot{V}(x) = -\frac{1}{2}\frac{g}{\ell}kx_1 \sin\, x_1 - kx_2^2$$

The term $x_1 \sin\, x_1 > 0$ for all $0 < |x_1| < \pi$. Taking $D = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$, we see that $\dot{V}(x) < 0$ over $D \setminus \{0\}$. Thus, by Lyapunov's Direct Method, we can conclude that $x = 0$ is asymptotically stable.

Let us consider the system

$$\dot{x} = -x^3$$

Now, let us analyze the stability properties of the equilibrium point $x = 0$ by using Lyapunov's direct method. The system here may be interpreted as a mechanical system where x is the velocity and a nonlinear friction acts on the system. No potential forces act on the system, so the system energy is the kinetic energy:

$$E = E_{\text{kin}} = \frac{1}{2}v^2 = \frac{1}{2}x^2$$

So, this is one motivation for this choice of Lyapunov function candidate $V(x) = \frac{1}{2}x^2$. An another motivation is that this is a simple choice of a quadratic Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$ where $P = I$ and since $x \in \mathbb{R}$, we have $V(x) = \frac{1}{2}x^2$.

Note that, $V \in \mathcal{C}^1$, $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$ which implies that V is positive definite in $D \setminus \{0\} = \mathbb{R} \setminus \{0\}$. The directional derivative reads $\dot{V}(x) = x\dot{x} = -x^4 < 0$, $\forall x \neq 0$ which tells us $\dot{V}$ is negative definite in $D \setminus \{0\} = \mathbb{R} \setminus \{0\}$. By Lyapunov's Direct Method, $x = 0$ is LAS. Note that, the conditions for being strict Lyapunov function are satisfied in the whole state space $\mathbb{R}$, so it is quite natural to as the following question:

Question: Can we conclude that the origin $x = 0$ is GAS?

Let us consider the following theorem!

Theorem (Lyapunov Theorem for GAS)

If

- $\exists$ *a* ***strict*** *Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$ *for* $x = 0$ *and*
- V *is radially unbounded*

then $x = 0$ *is globally asymptotically stable (GAS).*

Definition (Radial Unboundedness)

V is **radially unbounded** if and only if $V(x) \to \infty$ as $\|x\| \to \infty$.

Example

Turning back to $V(x) = \frac{1}{2}x^2 = \frac{1}{2}\|x\|^2$, this expression tells us that V is a radially unbounded function. This shows that by Lyapunov Theorem for GAS, we can conclude that $x = 0$ is GAS for $\dot{x} = -x^3$.

Question: Why the radial unboundedness condition is necessary to conclude global asymptotic stability based on Lyapunov analysis?

Lyapunov Theorem for Global Asymptotic Stability

For continuously differentiable fcns, say $V \in \mathcal{C}^1$, the following implications hold

- positive definiteness $\Rightarrow$ level surfaces are closed for small values of c, which is required for local results
- radial unboundedness $\Rightarrow$ level surfaces are closed $\forall c$, which is required for global results

So, if the level surfaces are not closed, we may have that $\|x\| \to \infty$ even if $\dot{V} < 0$.

Example

Let us take $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$.

Clearly, this function is positive definite. On the other hand,

- For $x_1 = 0$, $x_2 \to \infty \quad \Rightarrow \quad V(x) \to \infty$ as $\|x\| \to \infty$
- For $x_2 = 0$, $x_1 \to \infty \quad \Rightarrow \quad V(x) \to 1$ as $\|x\| \to \infty$!

So, $V(x)$ is not radially unbounded. There exist trajectories along which the time derivative of V is strictly negative, meaning that the trajectory intersects level curves corresponding to lower and lower c values, but the trajectory does not converge to the equilibrium point $x = 0$. See the figure on next slide!

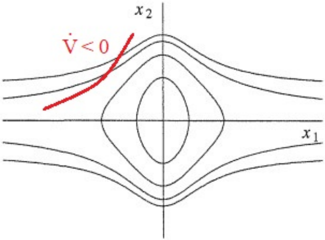


Figure: A Diverging Trajectory with $\dot{V}(x) < 0$.

Although the value of the V function decreases along the trajectory, the trajectory is allowed to slip away from the origin since the level curves are not closed.

See also KYP Lectures (L.4.4-10:57): `https://youtu.be/mIkgW_gUKjo?list=PLdeo5-jZaFjNPRGbKxWXrwnkNvjOkP_j8&t=657`

Lyapunov Theorem for Global Exponential Stability

We also have a Lyapunov theorem for exponential stability. We still consider the same system as before

$$\dot{x} = f(x)$$

where $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and $x = 0 \in D$ is an equilibrium point of the system.

Theorem (Exponential Stability)

If there exists a function $V : D \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ such that

i) $V \in C^1$

ii) $k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a, \forall x \in D$ *($V(x) \to \infty$ as $\|x\| \to \infty$)*

iii) $\dot{V}(x) \leq -k_3\|x\|^a, \quad \forall x \in D$

Then, $x = 0$ is ***exponentially stable (ES)****.*

Remark (Global Exponential Stability)

If the conditions in Exponential Stability Theorem are satisfied with $D = \mathbb{R}^n$, then $x = 0$ is globally exponentially stable (GES). The condition (ii) implies radial unboundedness condition. Hence, there is no need to impose radial unboundedness condition for GES.

Some further remarks:

- The Exponential Stability Theorem is also called Barbashin-Krasovskii Theorem.
- $\|\cdot\|$ can be any p-norm on the vector state space.
- This condition is stricter than the Asymptotic Stability Theorem because ES is stricter than AS.

Global Exponential Stability Convergence Rate: If the equilibrium point $x = 0$ of $\dot{x} = f(x)$ is globally exponentially stable, then the solution of the system satisfies

$$\|x(t)\| \leq \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(0)\| e^{-\frac{k_3}{k_2 a} t}, \quad \forall t \geq 0, \quad \|x(0)\| < c$$

where $c > 0$.

Example

Let us analyze the stability properties of the equilibrium point(s) of the system

$$\dot{x} = -x - x^3$$

by using Lyapunov direct method.

Note that,

$$\dot{x} = -x - x^3 = -x(1 + x^2) = 0$$

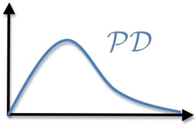
so that $x = 0$ is the only equilibrium point. As shown before, $V(x) = \frac{1}{2}x^2 = \frac{1}{2}\|x\|^2$ is a Lyapunov function candidate for all $x \in \mathbb{R}$ and $V \in \mathcal{C}^1$. (ii) of Exponential Stability Theorem is also satisfied with $k_1 = k_2 = \frac{1}{2}$, $a = 2$. The directional derivative of V along this system reads

$$\dot{V}(x) = x\dot{x} = -x^2 - x^4 \leq -x^2 = -\|x\|^2$$

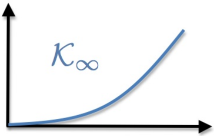
which tells us that (iii) of Exponential Stability Theorem is satisfied with $k_3 = 1$, $a = 2$. Note that $D = \mathbb{R}$, so that $x = 0$ is GES. The solution of this system satisfies the following GES convergence rate

$$\|x(t)\| \leq \|x(0)\|e^{-t}, \quad \forall t \geq 0.$$

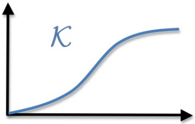
Comparison Functions



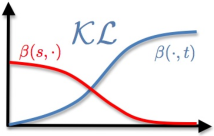
$$\begin{cases} \alpha \text{ continuous} \\ \alpha(0) = 0 \\ \alpha(s) > 0, \ \forall s > 0 \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{K} \\ \lim_{s\to\infty} \alpha(s) = \infty \end{cases}$$

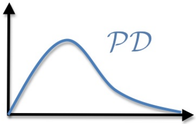


$$\begin{cases} \alpha \in \mathcal{PD} \\ \alpha \text{ increasing} \end{cases}$$



$$\begin{cases} \beta(\cdot,t) \in \mathcal{K}, \ \forall t \geq 0 \\ \beta(s,\cdot) \text{ nonincreasing}, \ \forall s \geq 0 \\ \lim_{t\to\infty} \beta(s,t) = 0, \ \forall s \geq 0 \end{cases}$$

Comparison Functions



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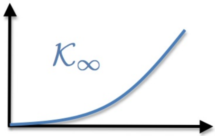

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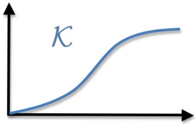

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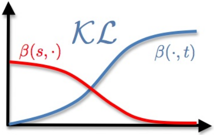
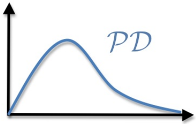


$$\begin{cases} \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0 \\ \beta(s, \cdot) \text{ nonincreasing},\ \forall s \geq 0 \\ \lim_{t\to\infty} \beta(s, t) = 0,\ \forall s \geq 0 \end{cases}$$



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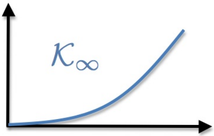

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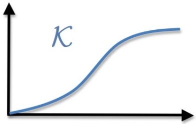

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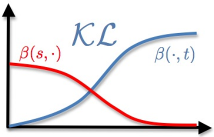

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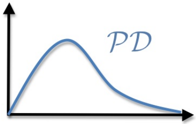
Comparison Functions



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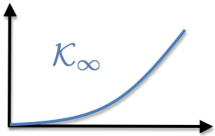

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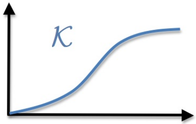

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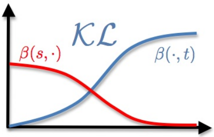

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Comparison Functions: Examples

Example

- $\alpha(s) = \frac{1}{1+s^c}$, for any $c > 0$
- $\alpha(s) = \frac{s^c}{1+s^c}$, for any $c > 0$
- $\alpha(s) = \tan^{-1}(s)$
- $\alpha(s) = \mathrm{sat}(s) = \begin{cases} s, & \text{if } |s| \leq 1 \\ \mathrm{sgn}(s), & \text{if } |s| > 1 \end{cases}$
- $\alpha(s) = \mathrm{sat}(s; a, b) = a + \frac{2(b-a)}{\pi} \tan^{-1}(s)$
- $\alpha(s) = s^c$, for any $c > 0$
- $\alpha(s) = \min\{s, s^2\}$
- $\beta(s, r) = \frac{s}{krs+1}$
- $\beta(s, r) = s^c e^{-r}$

Equivalent Representation of Radial Unboundedness

Lemma (4.3 in [Khalil, 2002])

Let $V : D \to \mathbb{R}$ be a continuous positive definite function (may not be $\mathcal{PD}$!) defined on $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r[0] \subset D$ for some $r > 0$. Then, there exist $\alpha_1, \alpha_2 \in \mathcal{K}$ defined on $[0, r]$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

for all $x \in B_r[0]$. If $D = \mathbb{R}^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold for all $x \in \mathbb{R}^n$. Moreover, if $V(x)$ is ***radially unbounded****, then $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.*

Example

- $V(x) = x^\top P x \implies \lambda_{\min}(P)|x|^2 \leq x^\top P x \leq \lambda_{\max}(P)|x|^2$

- Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.

Equivalent Representation of GAS

For a system with no inputs $\dot{x} = f(x)$, there is a well-known notion of global asymptotic stability (for short from now on, GAS, or "0-GAS" when referring to the system with no-inputs $\dot{x} = f(x, 0)$ associated to a given system with inputs $\dot{x} = f(x, u)$ due to Lyapunov, and usually defined in "ϵ-δ" terms. It is easy to show that the standard definition of GAS (0-GAS) in previous lecture is in fact equivalent to the existence of $\beta \in \mathcal{KL}$ satisfying the following, along the solutions of $\dot{x} = f(x)$ ($\dot{x} = f(x, 0)$)

$$|x(t, x_0)| \leq \beta(|x_0|, t), \qquad \forall x_0 \in \mathbb{R}^n, \ \forall t \geq 0.$$

Observe that, since β decreases on t, we have, in particular:

$$|x(t, x_0)| \leq \beta(|x_0|, 0), \qquad \forall x_0 \in \mathbb{R}^n, \ \forall t \geq 0.$$

which provides the stability part of the GAS definition while

$$|x(t, x_0)| \leq \beta(|x_0|, t) \underset{t \to \infty}{\to} 0, \qquad \forall x_0 \in \mathbb{R}^n,$$

which is the attractivity (convergence to the equilibrium point) part of the GAS definition.

Note: From now on, unless written explicitly, the solutions $x(t, x_0)$ or $x(t, x_0, u)$ for $\dot{x} = f(x)$ and $\dot{x} = f(x, u)$, respectively, will be written in short as $x(t)$ to avoid cumbersome notation!

Equivalent Lyapunov Theorem for Stability

Theorem (4.8 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(x) \leq W_2(x)$$
$$\frac{\partial V}{\partial x} f(x) \leq 0$$

for all $t \geq 0$ and $x \in D$, where W_1 and W_2 are continuous positive definite functions on D. Then, $x = 0$ is stable.

Equivalent Lyapunov Theorem for AS

Theorem (4.9 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(x) \leq W_2(x)$$
$$\frac{\partial V}{\partial x} f(x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where W_1, W_2 and W_3 are continuous positive definite functions on D. Then, $x = 0$ is asymptotically stable. Moreover, if r and c are chosen such that $B_r[0] = \{x \in D \mid |x| \leq r\}$ and $c < \min_{|x|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r[0] \mid W_2(x) \leq c\}$ satisfies

$$|x(t)| \leq \beta(|x(0)|, t), \quad \forall t \geq 0$$

for some $\beta \in \mathcal{KL}$.

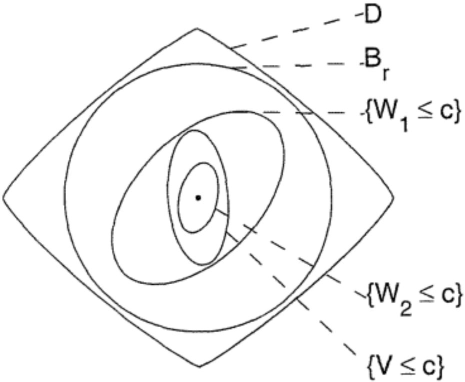

Figure: Geometric representation of sets in Theorem 4.9.

Equivalent Lyapunov Theorem for GAS

Theorem (4.9 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ ($D = \mathbb{R}^n$!) be a continuously differentiable function such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\frac{\partial V}{\partial x} f(x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and W_3 is continuous positive definite function on $\mathbb{R}^n$. Then, $x = 0$ is GAS.

Let us remember the Lyapunov theorem for ES/GES shown last week:

Theorem (4.10 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$k_1|x|^a \leq V(x) \leq k_2|x|^a$$
$$\frac{\partial V}{\partial x} f(x) \leq -k_3|x|^a$$

for all $t \geq 0$ and $x \in D$, where k_1, k_2, k_3 and a are positive constants. Then, $x = 0$ is ES. If $D = \mathbb{R}^n$, then $x = 0$ is GES.

Nonlinear systems: small inputs can induce big changes...

- Car trailer system
  - Video:
    https://www.youtube.com/watch?v=4jk9H5AB41M
- Aircraft
  - Video:
    https://www.youtube.com/watch?v=4UfmsqtTGa0
- Car active suspension system
  - Video:
    https://www.youtube.com/watch?v=kRt7H0k8A4k
- Building
  - Video:
    https://www.youtube.com/shorts/rJ72LruGgyU

For linear systems $\dot{x} = Ax + Bu$:

- $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$
- If A is a Hurwitz matrix ($Re(\lambda_i(A)) < 0$ for all $i = 1, ..., n$), then the linear system is 0-GAS.
- Such a 0-GAS linear system automatically satisfies all reasonable "input-to-state stability" properties [Sontag, 1990][1]:
  - Bounded inputs $\Rightarrow$ bounded state (BIBS) trajectories
  - Converging inputs $\Rightarrow$ converging state (CICS) trajectories

This is generally not the case for nonlinear systems $\dot{x} = f(x, u)$!

Example

Consider the scalar system ($n = 1$) with a single input ($m = 1$) $\dot{x} = -x + (x^2 + 1)u$:

- The system is clearly 0-GAS, since it reduces to $\dot{x} = -x$ when $u \equiv 0$.
- However, for $u = (2t + 2)^{-1/2}$ and $x_0 = \sqrt{2}$, the system produces unbounded and even diverging state trajectory $x(t) = (2t + 2)^{1/2}$!

[1] Mathematical Control Theory: Deterministic Finite Dimensional Systems

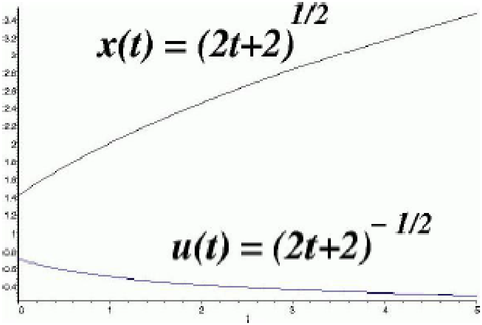

Figure: Diverging state for converging input.

Estimates (Gains) for Linear/Nonlinear Systems

Recall the solution of the linear system $\dot{x} = Ax + Bu$ can be written as:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

If A is Hurwitz, there exists some $k, \lambda > 0$ such that $\|e^{At}\| \leq ke^{-\lambda t}$ which, in turn, gives the following state estimate

$$\begin{aligned}
|x(t)| \leq& k|x(0)|e^{-\lambda t} + \int_0^t ke^{-\lambda(t-\tau)}\|B\||u(\tau)|d\tau \\
\leq& k|x(0)|e^{-\lambda t} + k\|B\| \sup_{\tau\in[0,t]} |u(\tau)|\, e^{-\lambda t} \int_0^t e^{\lambda\tau} d\tau \\
=& k|x(0)|e^{-\lambda t} + k\|B\| \sup_{\tau\in[0,t]} |u(\tau)|\, e^{-\lambda t} \left(\frac{e^{\lambda t} - 1}{\lambda}\right) \\
\leq& k|x(0)|e^{-\lambda t} + \frac{k\|B\|}{\lambda} \sup_{\tau\in[0,t]} |u(\tau)| \left(1 - e^{-\lambda t}\right) \\
\leq& \overline{k}|x(0)|e^{-\lambda t} + \overline{k} \sup_{\tau\in[0,t]} |u(\tau)|
\end{aligned}$$

where $\overline{k} = k \cdot \max\{1, \frac{\|B\|}{\lambda}\}$

Estimates (Gains) for Linear/Nonlinear Systems

Motivated with this estimation, for linear systems, three most typical ways of defining "input-to-state stability" in terms of operators $\{L^2, L^\infty\} \to \{L^2, L^\infty\}$ are as follows:

- "$L^\infty \to L^\infty$": $c|x(t)| \le |x_0|e^{-\lambda t} + \sup_{\tau\in[0,t]} |u(\tau)|$
- "$L^2 \to L^\infty$": $c|x(t)| \le |x_0|e^{-\lambda t} + \int_0^t |u(\tau)|^2 d\tau$
- "$L^2 \to L^2$": $c\int_0^t |x(\tau)|^2 d\tau \le |x_0| + \int_0^t |u(\tau)|^2 d\tau$

The missing case "$L^\infty \to L^2$" is less interesting, being too restrictive, for practical reasons! Concerning the nonlinear system $\dot{x} = f(x, u)$, in general, under "some" nonlinear coordinate change (see [Sontag, 2004]), we arrive to the following three concepts (or "estimates") for nonlinear systems:

- "$L^\infty \to L^\infty$": $\alpha(|x(t)|) \le \beta(|x_0|, t) + \sup_{\tau\in[0,t]} \gamma(|u(\tau)|)$
- "$L^2 \to L^\infty$": $\alpha(|x(t)|) \le \beta(|x_0|, t) + \int_0^t \gamma(|u(\tau)|)d\tau$
- "$L^2 \to L^2$": $\int_0^t \alpha(|x(\tau)|)d\tau \le \alpha_0(|x_0|) + \int_0^t \gamma(|u(\tau)|)d\tau$

Here, the functions (which measure the impacts of the state or input) are $\alpha, \alpha_0, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$. The "$L^\infty \to L^\infty$" (and "$L^2 \to L^2$" as well) estimate leads us to the first concept that of *input-to-state stability (ISS)* whereas "$L^2 \to L^\infty$" estimate leads us to the second concept that of *integral input-to-state stability (iISS)*.

Definition: Input-to-State Stability (ISS) (Sontag, IEEE TAC, 1989)

The system $\dot{x} = f(x, u)$ is ISS if there exist $\beta \in \mathcal{KL}$ and $\nu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu\left(\|u\|\right), \quad \forall t \geq 0.$$

$\rightarrow$ Vanishing transients "proportional" to initial state's norm
$\rightarrow$ Steady-state error "proportional" to input amplitude.

Definition: Integral Input-to-State Stability (iISS) (Sontag, SCL, 1998)

The system $\dot{x} = f(x, u)$ is iISS if there exist $\beta \in \mathcal{KL}$ and $\nu_1, \nu_2 \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu_1\left(\int_0^t \nu_2(|u(s)|)ds\right), \quad \forall t \geq 0.$$

$\rightarrow$ Measures the impact of input energy.

Definition: Input-to-State Stability (ISS) (Sontag, IEEE TAC, 1989)

The system $\dot{x} = f(x, u)$ is ISS if there exist $\beta \in \mathcal{KL}$ and $\nu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu\left(\|u\|\right), \quad \forall t \geq 0.$$

→ Vanishing transients “proportional” to initial state’s norm
→ Steady-state error “proportional” to input amplitude.

Definition: Integral Input-to-State Stability (iISS) (Sontag, SCL, 1998)

The system $\dot{x} = f(x, u)$ is iISS if there exist $\beta \in \mathcal{KL}$ and $\nu_1, \nu_2 \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu_1\left(\int_0^t \nu_2(|u(s)|)ds\right), \quad \forall t \geq 0.$$

→ Measures the impact of input energy.

ISS and iISS: Central tools in nonlinear analysis and control:

- Theoretical contributions to: output feedback, optimal control, hybrid systems, predictive control, chaotic systems. . .
- Applications in: robotics, production lines, transportation, bio-chemical networks, control under communication constraints, neuroscience, . . .

ISS	**iISS**
$\dot{x} = f(x,0)$ is GAS	$\dot{x} = f(x,0)$ is GAS
Bounded input $\Rightarrow$ Bounded state	Bounded energy input $\Rightarrow$ Bounded, converging state
Converging input $\Rightarrow$ Converging state	Converging input $\not\Rightarrow$ Converging state
Cascade: ISS + ISS $\Rightarrow$ ISS	Cascade: iISS + iISS $\not\Rightarrow$ iISS

In practice, some systems do exhibit robustness for inputs under a certain threshold, but diverge for larger ones.

Strong iISS: halfway between ISS and iISS.

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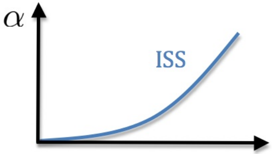
Strong iISS: halfway between ISS and iISS.

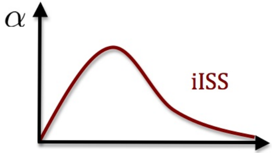
- Part of the success of ISS and iISS is due to their Lyapunov characterizations
- Lyapunov function candidate:
  - $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuously differentiable
  - $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$
  - $V(x) \to \infty$ whenever $|x| \to \infty$.

ISS and iISS characterization (Sontag, Wang, SCL, 1995 & Angeli et al., IEEE TAC, 2000)

The system $\dot{x} = f(x, u)$ is ISS (resp. iISS) if and only if there exist a Lyapunov function candidate V, $\gamma \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{K}_\infty$ (resp. $\alpha \in \mathcal{PD}$) such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(|x|) + \gamma(|u|).$$





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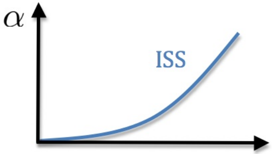

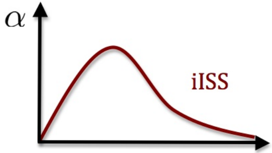

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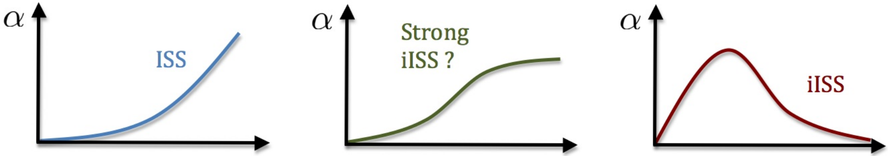

Definition: Strong iISS (Chaillet et. al., IEEE TAC, 2014)

The system $\dot{x} = f(x, u)$ is Strongly iISS if it is:

- iISS
- ISS with respect to small inputs

i.e., if there exist $\beta \in \mathcal{KL}$, $\nu_1, \nu_2, \nu \in \mathcal{K}_\infty$ and input threshold $R > 0$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu_1 \left(\int_0^t \nu_2(|u(s)|) ds \right)$$

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- For all $u \in \mathcal{U}$, the solution exists at all times
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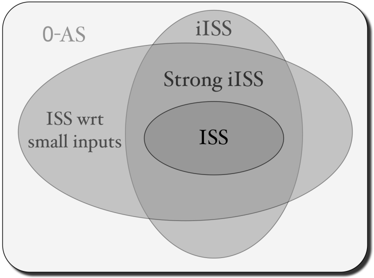

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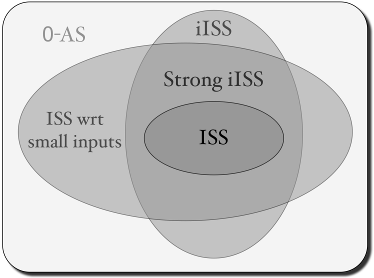

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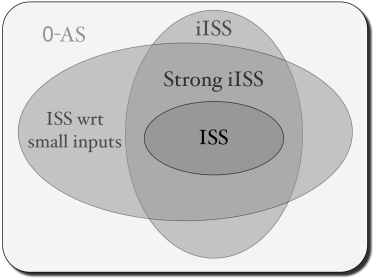

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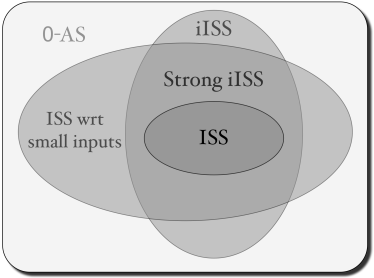

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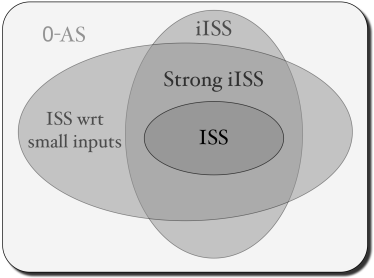

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Theorem: $\mathcal{K}$ dissipation rate $\Rightarrow$ Strong iISS (Chaillet et. al., IEEE TAC, 2014)

If there exists a Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha(|x|) + \gamma(|u|).$$

where $\alpha \in \mathcal{K}$ and $\gamma \in \mathcal{K}_\infty$, then the system $\dot{x} = f(x,u)$ is Strongly iISS with input threshold $R = \gamma^{-1} \circ \alpha(\infty)$.

Equivalently, we can state the following:

Corollary: Non-vanishing dissipation rate $\Rightarrow$ Strong iISS (Chaillet et. al., IEEE TAC, 2014)

If there exists a Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

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where $\gamma \in \mathcal{K}_\infty$ and W is continuous positive definite satisfying $W_\infty := \liminf_{|x|\to\infty} W(x) > 0$, then the system $\dot{x} = f(x,u)$ is Strongly iISS with input threshold $R = \gamma^{-1}(W_\infty)$.

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However, the converse does not hold:

Counter-example: Strong iISS $\not\Rightarrow \mathcal{K}$ dissipation rate (Chaillet et. al., IEEE TAC, 2014)

The scalar system

$$\dot{x} = -\frac{x}{1+x^2}\left[1 - |x|(u^2 - |u|)\right],$$

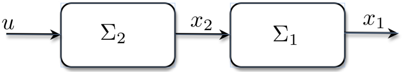
is Strongly iISS. However, for all $\alpha \in \mathcal{K}$ and $\gamma \in \mathcal{K}_\infty$ no differentiable function $V : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfies

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\alpha(|x|) + \gamma(|u|).$$

- iISS: $V_1(x) = \frac{1}{2}\ln(1+x^2)$ gives $\dot{V}_1 \leq -x^2/(1+x^2)^2 + u^2 + |u|$
- ISS wrt $|u| < 1$: $V_2(x) = x^4/4$ gives $\dot{V}_2 \leq -x^4/(1+x^2)$.

Preliminaries

Cascaded Systems



$$\Sigma_1 : \quad \dot{x}_1 = f_1(x_1, x_2)$$
$$\Sigma_2 : \quad \dot{x}_2 = f_2(x_2, u)$$

- ISS is naturally preserved in cascade (Sontag, EJC, 1995)
- iISS is not preserved by cascade (Panteley, Loría, Automatica, 2001 & Arcak et al., SICON, 2002).

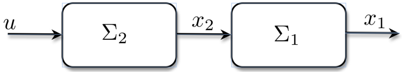
Theorem (Chaillet, Angeli, SCL, 2008)

Let V_1 and V_2 be two Lyapunov function candidates. Assume that there exist $\gamma_1, \gamma_2 \in \mathcal{K}$, and $\alpha_1, \alpha_2 \in \mathcal{PD}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) \leq -\alpha_1(|x_1|) + \gamma_1(|x_2|)$$
$$\frac{\partial V_2}{\partial x_2} f_2(x_2, u) \leq -\alpha_2(|x_2|) + \gamma_2(|u|) .$$

If $\gamma_1(s) = \mathcal{O}_{s\to 0^+}(\alpha_2(s))$, then the cascade is iISS.

- $q_2(s) = \mathcal{O}_{s\to 0^+}(q_1(s))$: Given $q_1, q_2 \in \mathcal{PD}$, we say that q_1 *has greater growth* than q_2 around zero if $\exists\, k \geq 0$ such that $\limsup_{s\to 0^+} q_2(s)/q_1(s) \leq k$.



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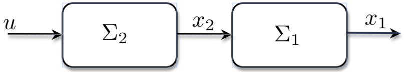
Cascaded Systems



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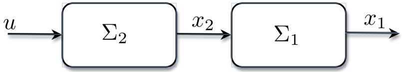
$$\frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) \leq -\alpha_1(|x_1|) + \gamma_1(|x_2|)$$
$$\frac{\partial V_2}{\partial x_2} f_2(x_2, u) \leq -\alpha_2(|x_2|) + \gamma_2(|u|)\,.$$

If $\gamma_1(s) = \mathcal{O}_{s \to 0^+}(\alpha_2(s))$, then the cascade is iISS.

- $q_2(s) = \mathcal{O}_{s \to 0^+}(q_1(s))$: Given $q_1, q_2 \in \mathcal{PD}$, we say that q_1 *has greater growth* than q_2 around zero if $\exists\, k \geq 0$ such that $\limsup_{s \to 0^+} q_2(s)/q_1(s) \leq k$.



$$\begin{aligned}\Sigma_1: \quad \dot{x}_1 &= f_1(x_1, x_2)\\ \Sigma_2: \quad \dot{x}_2 &= f_2(x_2, u)\end{aligned}$$

- ISS is naturally preserved in cascade (Sontag, EJC, 1995)
- iISS is not preserved by cascade (Panteley, Loría, Automatica, 2001 & Arcak et al., SICON, 2002).

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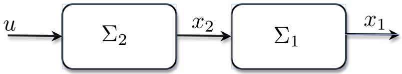

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Theorem: Strong iISS is preserved under cascade (Chaillet et. al., Automatica, 2014)

If the systems $\dot{x}_1 = f_1(x_1, u_1)$ and $\dot{x}_2 = f_2(x_2, u_2)$ are Strongly iISS, then the cascade is Strongly iISS.

Corollary: iISS + Strong iISS ⇒ iISS (Chaillet et. al., Automatica, 2014)

If $\dot{x}_1 = f_1(x_1, u_1)$ is Strongly iISS and $\dot{x}_2 = f_2(x_2, u_2)$ is iISS, then the cascade is iISS.

Corollary: GAS + Strong iISS ⇒ GAS (Chaillet et. al., Automatica, 2014)

If $\dot{x}_1 = f_1(x_1, u_1)$ is Strongly iISS and $\dot{x}_2 = f_2(x_2)$ is GAS, then

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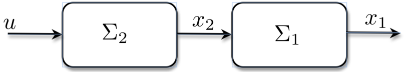

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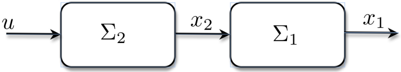

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	ISS	Strong iISS	iISS
0-GAS	✓	✓	✓
Forward completeness $\forall u \in \mathcal{U}$	✓	✓	✓
Bounded input-Bounded state	✓	For $\|u\| < R$	☹
Converging input-Converging state	✓	✓	☹
Preservation under cascade	✓	✓	Growth rate
Lyapunov characterization	$\alpha \in \mathcal{K}_\infty$	Open question	$\alpha \in \mathcal{PD}$

Time-Delay Systems (TDS)

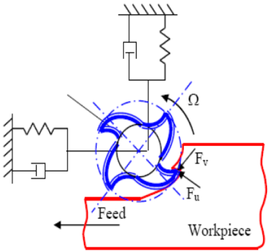


Figure: Rotating Milling Machine.

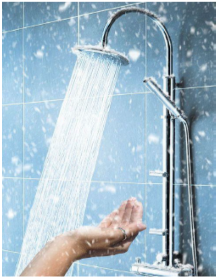


Figure: Shower.

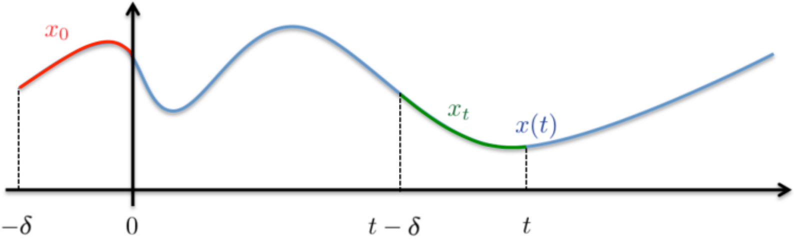
$$\dot{x}(t) = F(x(t)) + B(\omega t)(x(t) - x(t - \delta(t)))$$

$$\dot{x}(t) = -\alpha x(t - \delta), \ \alpha > 0$$

Consider the nonlinear TDS: $\dot{x}(t) = f(x_t, u(t))$

- State History: $x_t \in \mathcal{C}^n$ defined with the maximum delay $\delta \geq 0$ as

$$x_t(s) := x(t+s), \quad \forall s \in [-\delta, 0].$$



- $\mathcal{C}$: Set of all continuous functions $\varphi : [-\delta; 0] \to \mathbb{R}$.
- $\mathcal{U}$: Set of measurable essentially bounded signals to $\mathbb{R}^m$.
- Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.
- Given any $\phi \in \mathcal{C}^n$, $\|\phi\| := \sup_{\tau \in [-\delta, 0]} |\phi(\tau)|$.
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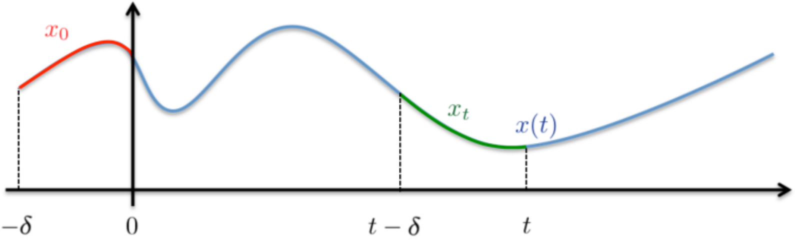

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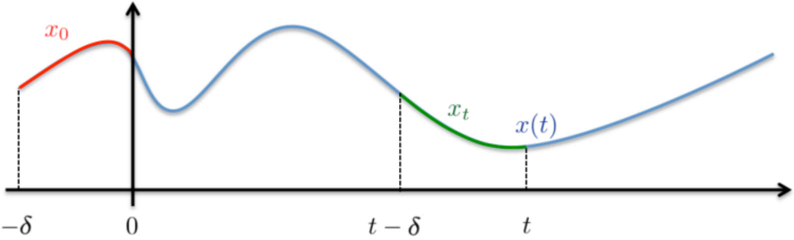

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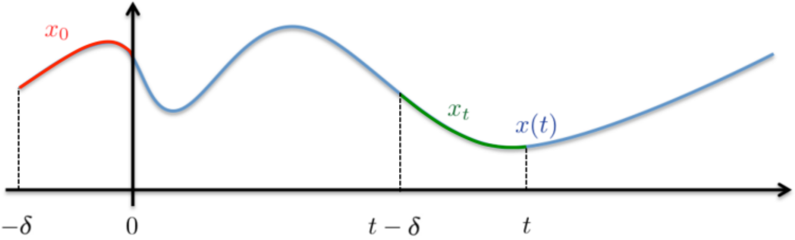

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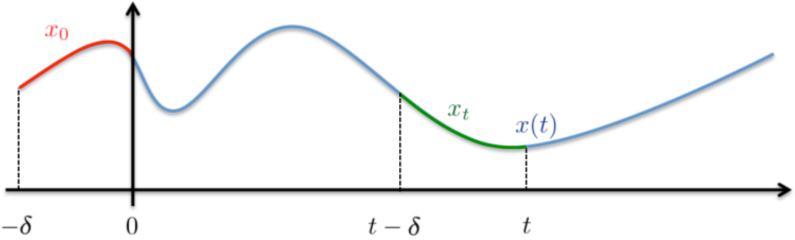

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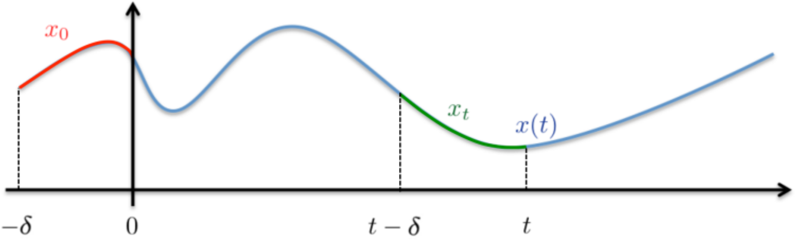

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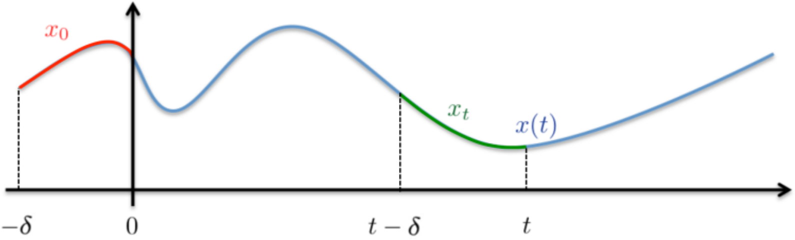

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Definition (0-GAS)

The TDS $\dot{x}(t) = f(x_t, u(t))$ is said to be globally asymptotically stable in the absence of inputs (0-GAS) (or the input-free system $\dot{x}(t) = f(x_t, 0)$ is GAS) if there exists $\beta \in \mathcal{KL}$ such that, the solution of the input-free system $\dot{x}(t) = f(x_t, 0)$ satisfies

$$|x(t)| \leq \beta(\|x_0\|, t), \qquad \forall t \geq 0.$$

Proposition (0-GAS characterization, (Hale, 1977, Corollary 3.1. , p. 119))

The TDS is 0-GAS if and only if there exist a LKF $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ and a function $\alpha \in \mathcal{PD}$ such that, for all $\phi \in \mathcal{C}^n$,

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The TDS $\dot{x}(t) = f(x_t, u(t))$ is said to be globally asymptotically stable in the absence of inputs (0-GAS) (or the input-free system $\dot{x}(t) = f(x_t, 0)$ is GAS) if there exists $\beta \in \mathcal{KL}$ such that, the solution of the input-free system $\dot{x}(t) = f(x_t, 0)$ satisfies

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- Forward completeness
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Proposition (ISS LKF, Necessity: (Pepe, Karafyllis, IJC, 2013), Sufficiency: (Pepe, Jiang, SCL, 2006))

The TDS is ISS if and only if there exists a LKF candidate $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{K}_\infty$ *and* $\gamma \in \mathcal{K}_\infty$*, such that the following holds:*

$$D^+ V(x_t, u(t)) \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \quad \forall t \geq 0.$$

$\rightarrow$ Finite-dimensional case: (Sontag, IEEE TAC, 1989).

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Definition (BEBS, BECS)

The TDS $\dot{x}(t) = f(x_t, u(t))$ is said to have the bounded energy-bounded state (BEBS) property, if there exists $\zeta \in \mathcal{K}_\infty$ such that its solution satisfies

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \sup_{t\geq 0} |x(t)| < \infty.$$

It is said to have the bounded energy-converging state (BECS) property if there exists $\zeta \in \mathcal{K}_\infty$ such that, its solution satisfies

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \lim_{t\to\infty} |x(t)| = 0.$$

Definition (UBEBS)

If the system $\dot{x}(t) = f(x_t, u(t))$ is said to have the uniform bounded energy-bounded state (UBEBS) property if there exist $\alpha, \xi, \zeta \in \mathcal{K}_\infty$ and $c \geq 0$ such that, $\forall x_0 \in \mathcal{C}^n$ and $\forall u \in \mathcal{U}$, its solution satisfies

$$\alpha(|x(t)|) \leq \xi(\|x_0\|) + \int_0^t \zeta(|u(s)|)ds + c, \quad \forall t \geq 0.$$

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Proposition (iISS⇔0-GAS+UBEBS, (Chaillet, G, Pepe, IEEE TAC, 2022))

The TDS $\dot{x}(t) = f(x_t, u(t))$ is iISS if and only if it is 0-GAS and owns the UBEBS property.

Lemma (UBEBS with $c = 0$, (Chaillet, G, Pepe, IEEE TAC, 2022))

If the system $\dot{x}(t) = f(x_t, u(t))$ is 0-GAS, then the following properties are equivalent:

- *The system satisfies the UBEBS estimate.*
- *The system satisfies the UBEBS estimate with $c = 0$.*

Proposition (iISS⇔0-GAS+zero-output dissipativity, (Chaillet, G, Pepe, IEEE TAC, 2022))

The TDS $\dot{x}(t) = f(x_t, u(t))$ is iISS if and only if it is 0-GAS and there exists a LKF $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ and $\mu \in \mathcal{K}_\infty$ such that

$$D^+V(\phi, v) \leq \mu(|v|), \quad \forall \phi \in \mathcal{C}^n, \forall v \in \mathbb{R}^m.$$

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- an iISS LKF with $\mathcal{KL}$ dissipation rate for $\dot{x}(t) = f(x_t, u(t))$ if $\exists \sigma \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall \phi \in \mathcal{C}^n$ and $\forall v \in \mathbb{R}^m$,
$$D^+ V(\phi, v) \leq -\sigma(|\phi(0)|, \|\phi\|) + \gamma(|v|).$$

- α and σ are called dissipation rate,
- γ is called supply rate.

Theorem (iISS LKF Characterizations, (Chaillet, G, Pepe, IEEE TAC, 2022))

The following statements are equivalent for the TDS $\dot{x}(t) = f(x_t, u(t))$:

(i) *The TDS admits a coercive iISS LKF with history-wise dissipation.*

(ii) *The TDS admits an iISS LKF with LKF-wise dissipation.*

(iii) *The TDS admits an iISS LKF with history-wise dissipation.*

(iv) *The TDS admits an iISS LKF with $\mathcal{KL}$ dissipation.*

(v) *The TDS is iISS.*

Moreover, the TDS is iISS if

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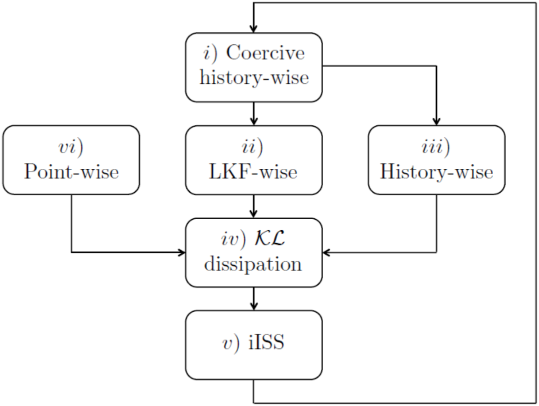

Figure: Proof Strategy.

Proof (Sketch).

- (iv)⇒(v): iISS LKF with $\mathcal{KL}$ dissipation ⇒ 0-GAS+zero-output dissipativity ⇒ iISS.
- (v)⇒(i): iISS ⇒
  - ∃ coercive LKF $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\nu \in \mathcal{K}_\infty$ with $D^+V(\phi, v) \leq \nu(|v|)$, $\forall \phi \in \mathcal{C}^n$, $v \in \mathbb{R}^m$ (Lin, Wang, CDC, 2018).
  - ∃ coercive LKF $V_1 : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\pi \in \mathcal{K} \cap \mathcal{C}^1$ with $\pi'(s) > 0$, $\forall s \geq 0$, $\alpha \in \mathcal{PD}$, $\gamma \in \mathcal{K}_\infty$ such that $W_1 := \pi \circ V_1$ satisfies $D^+W_1(\phi, v) \leq -\alpha(\|\phi\|) + \gamma(|v|)$.

  $\mathcal{V} := V + W_1$ is a coercive iISS LKF with history-wise dissipation.
- (i)⇒(iii): Trivial as any coercive LKF is a LKF.

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- (iv)⇒(v): iISS LKF with $\mathcal{KL}$ dissipation ⇒ 0-GAS+zero-output dissipativity ⇒ iISS.
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Proof (Sketch-Continued).

Fact (Angeli et. al, IEEE TAC, 2000): $\forall \alpha \in \mathcal{PD}$, $\exists \mu \in \mathcal{K}_\infty, \ell \in \mathcal{L}$ such that $\alpha(s) \geq \mu(s)\ell(s)$, $\forall s \geq 0$.

- (i)⇒(ii): V is coercive history-wise LKF $\Rightarrow \exists V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{PD}$, $\underline{\alpha}, \overline{\alpha}, \gamma \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|)$$

$$\begin{aligned} D^+ V(\phi, v) &\leq -\alpha(\|\phi\|) + \gamma(|v|) \\ &\leq -\mu(\|\phi\|)\ell(\|\phi\|) + \gamma(|v|) \\ &\leq -\mu \circ \overline{\alpha}^{-1}(V(\phi))\ell \circ \underline{\alpha}^{-1}(V(\phi)) + \gamma(|v|) \end{aligned}$$

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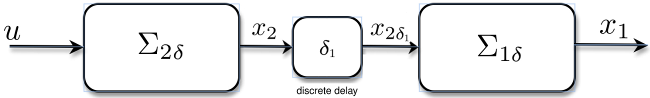


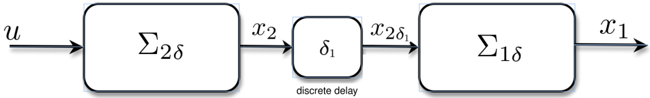
Consider two nonlinear TDS in cascade:

$$\begin{aligned} \Sigma_{1\delta} &: \quad \dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1)) \\ \Sigma_{2\delta} &: \quad \dot{x}_2(t) = f_2(x_{2t}, u(t)) \end{aligned}$$

$\rightarrow \delta_1 \in [0, \delta]$: Interconnection through discrete delay.

Questions:

- iISS preserved under cascade interconnected TDS?
- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and UBEBS?



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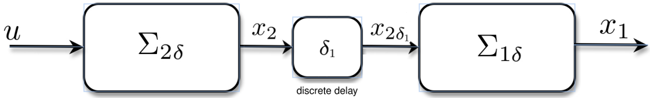

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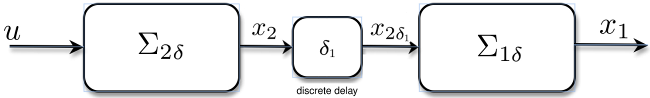

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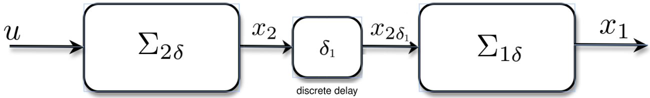

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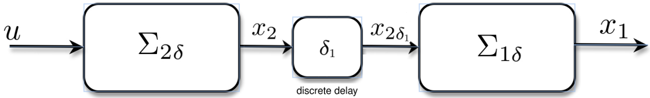

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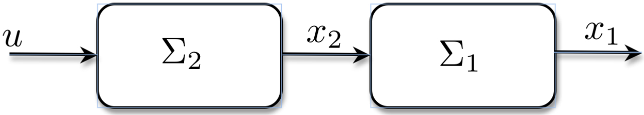

Consider two nonlinear systems in cascade:

$$\Sigma_1 : \quad \dot{x}_1 = f_1(x_1, x_2)$$
$$\Sigma_2 : \quad \dot{x}_2 = f_2(x_2, u)$$

- ISS is naturally preserved in cascade (Sontag, EJC, 1995)
- iISS is not preserved by cascade (Panteley, Loría, Automatica, 2001 & Arcak et al., SICON, 2002).

Questions:

- ~~iISS preserved under cascade interconnected TDS?~~
- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and UBEBS?

Theorem (Chaillet, Angeli, SCL, 2008)

Let V_1 and V_2 be two Lyapunov function candidates. Assume that there exist $\gamma_1, \gamma_2 \in \mathcal{K}$, and $\alpha_1, \alpha_2 \in \mathcal{PD}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $u \in \mathbb{R}^m$,

$$\begin{aligned} \frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) &\leq -\alpha_1(|x_1|) + \gamma_1(|x_2|) \\ \frac{\partial V_2}{\partial x_2} f_2(x_2, u) &\leq -\alpha_2(|x_2|) + \gamma_2(|u|) \,. \end{aligned}$$

If $\gamma_1(s) = \mathcal{O}_{s\to 0^+}(\alpha_2(s))$, then the cascade is iISS.

$\to q_2(s) = \mathcal{O}_{s\to 0^+}(q_1(s))$: Given $q_1, q_2 \in \mathcal{PD}$, we say that q_1 *has greater growth* than q_2 around zero if $\exists\, k \geq 0$ such that $\limsup_{s\to 0^+} q_2(s)/q_1(s) \leq k$.

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} Above condition valid for TDS?

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- Conditions to ensure 0-GAS and BEBS?

} Above condition valid for TDS?

Theorem (G, Chaillet, Automatica, 2022)

Assume that $\exists$ *two LKF candidates* $V_i : \mathcal{C}^{n_i} \to \mathbb{R}_{\geq 0}$, $\sigma_i \in \mathcal{KL}$ *and* $\gamma_i \in \mathcal{K}_\infty$, $i \in \{1, 2\}$, *such that,* $\forall \phi \in \mathcal{C}^{n_1}$, $v_1 \in \mathbb{R}^{n_2}$

$$D^+ V_1(\phi, v_1) \leq -\sigma_1(|\phi(0)|, \|\phi\|) + \gamma_1(|v_1|), \tag{D1}$$

and, $\forall \varphi \in \mathcal{C}^{n_2}$, $v \in \mathbb{R}^m$

$$D^+ V_2(\varphi, v) \leq -\sigma_2(|\varphi(0)|, \|\varphi\|) + \gamma_2(|v|) \tag{D2}$$

for all $t \geq 0$. *Assume further that the following holds:*

$$\gamma_1(s) = \mathcal{O}_{s \to 0^+}(\sigma_2(s, 0)). \tag{GR}$$

Then, the cascade is iISS.

Lemma (G, Chaillet, Automatica, 2022)

Let $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ *be a LKF candidate satisfying, for all* $\phi \in \mathcal{C}^n$,

$$D^+ V(\phi) \leq -\sigma(|\phi(0)|, \|\phi\|),$$

for some $\alpha \in \mathcal{PD}$ *and* $\eta \in \mathcal{K}_\infty$. *Let* $\tilde{\alpha} \in \mathcal{PD}$ *satisfying*

$$\tilde{\alpha}(s) = \mathcal{O}_{s \to 0^+}(\sigma(s, 0)).$$

Then, $\exists$ *a continuously differentiable function* $\rho \in \mathcal{K}_\infty$ *such that the functional* $\tilde{V} := \rho \circ V$ *satisfies*

$$D^+ \tilde{V}(\phi) \leq -\tilde{\alpha}(|\phi(0)|).$$

- Proof can be made applying chain rule to $\tilde{V} := \rho \circ V$.
- Result in finite-dimension: (Sontag, Teel, TAC, 1995)

Proof of Theorem (Sketch).

Proof of Forward Completeness.

- (D2) implies forward completeness of $\dot{x}_2(t) = f_2(x_{2t}, u(t))$.
- (D1) with $u_1(t) = x_2(t - \delta_1) \Rightarrow \nexists$ any finite escape time for $x_1(t)$.

Proof of 0-GAS (Sketch).

- Consider the input-free system

$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t - \delta_1)),$$
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- (GR)+Lemma $\Rightarrow \exists \rho \in \mathcal{K}_\infty \cap \mathcal{C}^1$ such that $\tilde{V}_2 := \rho \circ V_2$ satisfies

$$D^+ \tilde{V}_2(x_{2t}) \leq -2\gamma_1(|x_2(t)|). \quad (1)$$

- Now, consider the LKF defined as

$$\mathcal{V}_2(\phi_2) := \tilde{V}_2(\phi_2) + \int_{-\delta_1}^{0} \gamma_1(|\phi_2(\tau)|)d\tau, \quad \forall \phi_2 \in \mathcal{C}^{n_2}.$$

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Proof of Theorem (Continued).

Proof of Theorem: 0-GAS (Sketch-Continued).

- In view of (1), its Dini derivative therefore reads

$$D^{+}\mathcal{V}_2(x_{2t}) \leq -\gamma_1(|x_2(t)|) - \gamma_1(|x_2(t-\delta_1)|). \quad (2)$$

- Furthermore (D1) ensures that

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$$D^{+}\mathcal{V}(x_t) \leq -\frac{\alpha_1(|x_1(t)|) + \gamma_1(|x_2(t)|)}{1+\eta_1(\mathcal{V}(x_t))},$$

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Proof of UBEBS (Sketch).

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- (GR)+"Fact" (Angeli et. al., IEEE TAC, 2000) $\Rightarrow \exists k \ge 0$ such that $\alpha_2(s) > k\gamma_1(s)$, $\forall s \in [0,1]$.
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$$\int_0^t \gamma_1(|x_2(\tau)|)d\tau \le \xi_1(\|x_{10}\|) + \xi_2\left(\int_0^t \gamma_2(|u(\tau)|)d\tau\right)$$
for some $\xi_1, \xi_2 \in \mathcal{K}_\infty$ which plays the key role to get UBEBS with $c = 0$.

$\therefore$ **0-GAS + UBEBS with $c = 0 \Rightarrow$ iISS.** □

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for some $\xi_1, \xi_2 \in \mathcal{K}_\infty$ which plays the key role to get UBEBS with $c = 0$.

$\therefore$ **0-GAS + UBEBS with $c = 0 \Rightarrow$ iISS.** □

Proof of Theorem (Continued).

Proof of UBEBS (Sketch).

- (D1)+"Fact" (Angeli et. al., IEEE TAC, 2000) $\Rightarrow \exists \alpha_1 \in \mathcal{K}$, $\eta_1 \in \mathcal{K}_\infty$ such that,
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Consider the following input-free cascade:

$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1))$$
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Corollary

Assume that $\exists$ *two LKF candidates* $V_1 : \mathcal{C}^{n_1} \to \mathbb{R}_{\geq 0}$ *and* $V_2 : \mathcal{C}^{n_2} \to \mathbb{R}_{\geq 0}$, $\sigma_1, \sigma_2 \in \mathcal{KL}$ *and* $\gamma_1 \in \mathcal{K}_\infty$ *such that,* $\forall \phi \in \mathcal{C}^{n_1}$, $v_1 \in \mathbb{R}^{n_2}$

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and, $\forall \varphi \in \mathcal{C}^{n_2}$,

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Example

Consider the following TDS:

$$\dot{x}(t) = -\frac{|x(t)|}{1 + \|x_t\|^2} + u(t).$$

Consider the LKF (proposed in Pepe, Jiang, SCL, 2006) defined as

$$W(\phi) := \sup_{s\in[-\delta,0]} e^s Q(\phi(s)), \quad \forall \phi \in \mathcal{C},$$

where the function $Q : \mathbb{R} \to \mathbb{R}^+$ is defined as

$$Q(x) = \begin{cases} \frac{1}{2}x^2, & if|x| \leq 1, \\ |x| - \frac{1}{2}, & if|x| > 1. \end{cases}$$

After cumbersome calculation, it is possible to get

$$D^+W(\phi, v) \leq \begin{cases} -W(\phi), & \text{if } W > Q(\phi(0)), \\ \max\left\{-W, Q'(\phi(0))\left(\frac{-\phi(0)}{1+\|\phi\|^2} + v\right)\right\}, & \text{if } W = Q(\phi(0)). \end{cases}$$

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On contrary $V(\phi) = |\phi(0)|$ satisfies, $\forall \phi \in \mathcal{C}$ and $\forall v \in \mathbb{R}$

$$D^+ V(\phi, v) \leq -\frac{|x(t)|}{1 + \|x_t\|^2} + |v|,$$

does the same job with $\mathcal{KL}$ dissipation.

Example

Consider the following cascade TDS:

$$\dot{x}_1(t) = -\,\mathrm{sat}(x_1(t)) + \frac{1}{4}\mathrm{sat}(x_1(t-1)) + x_1(t)x_2(t-2)^2 \qquad \text{(C1a)}$$

$$\dot{x}_2(t) = -\frac{3}{2}x_2(t) + x_2(t-1) + u(t)\int_{t-1}^{t} x_2(\tau)d\tau. \qquad \text{(C1b)}$$

- $\mathrm{sat}(s) := \mathrm{sign}(s)\min\{|s|, 1\}$ for all $s \in \mathbb{R}$.
- $n_1 = n_2 = 1$, $m = 1$, $\delta_1 = \delta = 2$.

Consider the LKF candidates defined as

$$V_1(\phi_1) := \ln\left(1+\phi_1(0)^2 + \frac{1}{2}\int_{-1}^{0}\phi_1(\tau)\mathrm{sat}(\phi_1(\tau))d\tau\right),$$

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By deriving, we have

$$D^+ V_1(x_{1t}, x_{2t}) \le -\frac{x_1(t)\mathrm{sat}(x_1(t))}{1+\eta_1(\|x_{1t}\|)} + 2x_2(t-2)^2,$$

$$D^+ V_2(x_{2t}, u(t)) \le -\frac{x_2(t)^2}{1+\eta_2(\|x_{2t}\|)} + |u(t)|.$$

where $\eta_i(s) = e^{\overline{\alpha}_i(s)} - 1$, $i = 1, 2$. The functions are

- $\alpha_1(s) = \mathrm{sat}(s)s$,
- $\alpha_2(s) = s^2$,
- $\gamma_1(s) = 2s^2$ and
- $\gamma_2(s) = s$.

$\rightarrow$ Growth-rate condition: $2s^2 = \mathcal{O}_{s\to 0^+}(s^2)$.

$\therefore$ Thus, the cascade (C1) is iISS.

Illustrative Examples

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Illustrative Examples

Example

Consider the bilinear TDS in cascade with a discrete-delayed driven and a distributed-delayed driving subsystems

$$\dot{x}(t) = A_1 x(t) + \left(\sum_{i=1}^{n_2} z_i(t) A_{1,i} \right) x(t-\delta) + B_1 z(t), \tag{C2a}$$

$$\dot{z}(t) = A_2 z(t) + \left(\sum_{i=1}^{m} u_i(t) A_{2,i} \right) \int_{-\delta}^{0} z(t+s) ds + B_2 u(t). \tag{C2b}$$

- A_1 and A_2 are Hurwitz matrices.
- All matrices have appropriate dimensions.
- We know from (Pepe, Jiang, SCL, 2006) that (C2a) and (C2b) are iISS but not ISS.

We consider the LKF candidates, for all $\phi \in \mathcal{C}^{n_1}$ and $\varphi \in \mathcal{C}^{n_2}$, as

$$V_1(\phi) = \ln \left(1 + \phi^\top(0) P_1 \phi(0) + p_2 \int_{-\delta}^{0} |\phi(s)|^2 ds \right),$$

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where P_1 and R_1 are positive definite symmetric matrices and $p_1 > 0$ and $r_1 > 0$ are scalars to be determined later.

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The derivative of the LKFs along the lines of the subsystems reads

$$D^+ V_1(x_t, z(t)) \leq -\frac{|x(t)|^2}{1 + \eta_1(\|x_t\|)} + \gamma_1 |z(t)|^2$$

$$D^+ V_2(z_t, u(t)) \leq -\frac{|z(t)|^2}{1 + \eta_2(\|z_t\|)} + \gamma_2 |u(t)|^2$$

for $\eta_i(s) = e^{\overline{\alpha}_i(s)} - 1$, $i = 1, 2$ and appropriately chosen $\gamma_1, \gamma_2 > 0$.

$\therefore$ GR condition is also satisfied. Thus, the cascade (C2) is iISS.

Example

Consider the bilinear TDS in cascade with a discrete-delayed driven and a distributed-delayed driving subsystems

$$\dot{x}(t) = A_1 x(t) + \left(\sum_{i=1}^{n_2} z_i(t) A_{1,i} \right) x(t-\delta) + B_1 z(t), \tag{C2a}$$

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$$D^+ V_1(x_t, z(t)) \leq - \frac{|x(t)|^2}{1 + \eta_1(\|x_t\|)} + \gamma_1 |z(t)|^2$$

$$D^+ V_2(z_t, u(t)) \leq - \frac{|z(t)|^2}{1 + \eta_2(\|z_t\|)} + \gamma_2 |u(t)|^2$$

for $\eta_i(s) = e^{\overline{\alpha}_i(s)} - 1$, $i = 1, 2$ and appropriately chosen $\gamma_1, \gamma_2 > 0$.

$\therefore$ GR condition is also satisfied. Thus, the cascade (C2) is iISS.

Cascade Interconnected iISS TDS: Illustrative Examples

Example

Consider the bilinear TDS in cascade with a discrete-delayed driven and a distributed-delayed driving subsystems

$$\dot{x}(t) = A_1 x(t) + \left(\sum_{i=1}^{n_2} z_i(t) A_{1,i}\right) x(t-\delta) + B_1 z(t), \quad \text{(C2a)}$$

$$\dot{z}(t) = A_2 z(t) + \left(\sum_{i=1}^{m} u_i(t) A_{2,i}\right) \int_{-\delta}^{0} z(t+s)ds + B_2 u(t). \quad \text{(C2b)}$$

The derivative of the LKFs along the lines of the subsystems reads

$$D^+ V_1(x_t, z(t)) \leq -\frac{|x(t)|^2}{1+\eta_1(\|x_t\|)} + \gamma_1 |z(t)|^2$$

$$D^+ V_2(z_t, u(t)) \leq -\frac{|z(t)|^2}{1+\eta_2(\|z_t\|)} + \gamma_2 |u(t)|^2$$

for $\eta_i(s) = e^{\overline{\alpha}_i(s)} - 1$, $i = 1, 2$ and appropriately chosen $\gamma_1, \gamma_2 > 0$.

$\therefore$ GR condition is also satisfied. Thus, the cascade (C2) is iISS.

Conclusions and Perspectives

Overview:

- Several LK characterizations of iISS for TDS.
- $\mathcal{KL}$ dissipation rates simplify the iISS analysis.
- The existing theory for input/disturbance-free systems is also relaxed.
- Robustness Property: 0-GAS+UBEBS$\Leftrightarrow$iISS.
- Conditions under which the cascade of two iISS TDS is iISS.
- Growth restrictions on the input rate of the driven subsystem and the dissipation rate of the driving one.

Open Questions:

- Converse theorem for iISS LKF with point-wise dissipation.
- Characterizations for ISS TDS.
- Converse theorem for Strong iISS in finite-dimensional context.
- Solns based characterizations for iISS TDS.
- Conditions to ensure strong iISS for TDS.

} Allah'ım konuyu biliyorsun...

- For further open questions see [Chaillet, Karafyllis, Pepe, Wang, MCSS, 2022, Chapter 8].

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