MTM3502-Partial Differential Equations

Gökhan Göksu, PhD

Week 1

Contact Information and Course Evaluation Criteria

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- Term Evaluation (Midterm): 60 %
- End of Term Evaluation (Final Exam): 40 %
- Office Hours: Friday, 14:00-17:00 (and/or by appointment)
- Attendance will not affect your grade!

Course Content and References

- Introduction: Basic Concepts and Definitions
- First Order PDEs
- Second Order Linear PDEs
- The Cauchy Problem
- The Seperation of Variables
- Cauchy Problems for Wave, Heat and Laplace's Eqns

- J. N. Sharma and K. Singh, Partial Differential Equations for Engineers and Scientist, Alpha Science, 2000.
- T. Myint-U and L. Debnath, Linear Partial Differential Equations for Scientists and Engineers, Birkhäuser, 2007.
- K. Koca, Kısmi Türevli Denklemler, Gazi Kitabevi, 2008.

Differential equation (DE) is an equation which contains and unknown function of one or several variables and its derivatives with respect to its arguments. The study of DEs is a crucial and fundamental subject in applied mathematics and plays an important role of various disciplines such as engineering, physics, mechanics etc...

DEs are generally classified into two categories: ordinary differential equations (ODEs), which are DEs of an unknown function depending on a single independent variable and partial differential equations (PDEs), which are DEs of an unknown function depending on two or more variables.

The general form of a PDE for an unknown function $u(x_1, x_2, ..., x_n)$ is

$$F(x_1, x_2, ..., x_n, u, u_{x_1}, u_{x_2}, ..., u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, ...) = 0 \qquad (1.1)$$

where $(x_1, x_2, ..., x_n)$ are independent variables, u_{x_i} denotes the partial derivative $\frac{\partial u}{\partial x_i}$ for $i = 1, 2, ..., n$ and F is an appropriately defined nonlinear function.

If a PDE contains n^{th} order derivatives of the unknown function and if n is the highest order partial derivative, the PDE is called an n^{th} order PDE. For instance, a third order PDE is of the general form

$$F(x_1, x_2, x_3, u, u_{x_1}, u_{x_2}, u_{x_3}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_1 x_3}, ...) = 0. \qquad (1.2)$$

Introduction: Basic Concepts and Definitions

To represent a physical phenomena the independent variables are taken as (t, x, y) to demonstrate 2+1 spatiotemporal (1 temporal and 2 spatial variables) representation. In this case the general form will be

$$F(t, x, y, u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, ...) = 0. \tag{1.3}$$

A few well-known examples of PDEs are as the following:

- Laplace equation:

$$u_{xx} + u_{yy} = 0 \tag{1.4}$$

- Heat conduction equation:

$$u_{xx} + u_{yy} = \frac{1}{K} u_t \tag{1.5}$$

- Wave equation:

$$u_{xx} + u_{yy} = \frac{1}{C^2} u_{tt} \tag{1.6}$$

- Burgers' equation:

$$\mu u_{xx} - u u_x = u_t \tag{1.7}$$

Structural Classification of First Order PDEs

According to their linearity, first order PDEs are classified as follows:

- First order linear PDEs,
- First order semi-linear PDEs,
- First order quasi-linear PDEs and
- First order nonlinear PDEs.

In this subsection, our unknown function is taken of the form $u = u(x, y)$, which is a function of x and y.

Structural Classification of First Order PDEs: FO Linear PDEs

A PDE is called a first order linear PDE, if the defining function F is a linear function of u, u_x and u_y:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \tag{1.8}$$

where a, b, c and d are appropriately defined nonlinear functions of x and y. The characteristic equations for (1.8) are

$$\begin{aligned} \frac{dx}{dt} &= a(x,y), \\ \frac{dy}{dt} &= b(x,y), \\ \frac{du}{dt} + c(x,y)u &= d(x,y). \end{aligned} \tag{1.9}$$

where t defines the changes along a characteristic line.

A PDE is called a first order semi-linear PDE, if it has the form:

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \qquad (1.10)$$

where a and b are functions of x and y and c is a function of x, y and u. The characteristic equations for (1.10) are

$$\begin{aligned} \frac{dx}{dt} &= a(x,y), \\ \frac{dy}{dt} &= b(x,y), \\ \frac{du}{dt} &= c(x,y,u). \end{aligned} \qquad (1.11)$$

Structural Classification of First Order PDEs: FO Quasi-Linear PDEs

A PDE is called a first order quasi-linear PDE, if it has the form:

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{1.12}$$

where a, b and c are functions of x, y and u. The characteristic equations for (1.12) are

$$\begin{aligned} \frac{dx}{dt} &= a(x,y,u), \\ \frac{dy}{dt} &= b(x,y,u), \\ \frac{du}{dt} &= c(x,y,u). \end{aligned} \tag{1.13}$$

Structural Classification of First Order PDEs: FO Nonlinear PDEs

A PDE is called a first order nonlinear PDE, if the defining function F is a nonlinear function of x, y, u_x, u_y and u:

$$F(x, y, u, u_x, u_y) = 0. \tag{1.14}$$

The characteristic equations for a first order nonlinear PDE are

$$\begin{aligned} \frac{dx}{dt} &= F_p, \ \frac{dy}{dt} = F_q, \ \frac{du}{dt} = pF_p + qF_q, \\ \frac{dp}{dt} &= -F_x - pF_u, \ \frac{dq}{dt} = -F_y - qF_u, \end{aligned} \tag{1.15}$$

where

$$p := u_x = \frac{\partial u}{\partial x}, \ q := u_y = \frac{\partial u}{\partial y}. \tag{1.16}$$

The first order nonlinear PDE (1.14) can also be reformulated with (1.16)

$$F(x, y, u, p, q) = 0. \tag{1.17}$$

The method of characteristics for these type of PDEs will be demonstrated later in detail as a solution methodology namely as “Cauchy’s Method of Characteristics".

Formulation of First Order PDEs

Consider the relation

$$\phi(x, y, u) = 0. \tag{1.18}$$

Differentiating (1.18) with respect to x and y, respectively, we get

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial u} = 0, \tag{1.19}$$

$$\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial u} = 0. \tag{1.20}$$

The set of equations (1.18)-(1.20) constitute three equations involving p and q. It is possible to write the equation (1.18) of the form (1.17), which represents the system of surfaces in space gives rise to a PDE of the first order.

Example 1

Let us consider the following equation

$$2u = (ax + y)^2 + b \tag{1.21}$$

where a and b are nonzero arbitrary constants. Find the PDE of least order from the given family of surfaces by eliminating the arbitrary constants.

Example 2

Consider the following equation

$$u = xy + \varphi(x^2 + y^2) \tag{1.25}$$

where φ is any differentiable function. Find the PDE of least order from the given family of surfaces by eliminating the arbitrary function φ.

Example 3

Consider the relation

$$\varphi(x^2 + y^2 + u^2, u^2 - 2xy) = 0 \tag{1.29}$$

where φ is any partially differentiable function. Find the PDE of least order from the given family of surfaces by eliminating the arbitrary function φ.