MTM3502-Partial Differential Equations

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Week 10

In parabolic PDEs, since $B^2 - 4AC = 0$, the determinant yields to $B^{*2} - 4A^*C^* = 0$. We have two cases to obtain canonical forms of hyperbolic PDEs:

- $A^* = B^* = 0$ and $C^* \neq 0$,
- $A^* \neq 0$ and $B^* = C^* = 0$.

The *first canonical form of the parabolic PDEs*, considering $A^* \neq 0$ and $B^* = C^* = 0$, is

$$u_{\xi\xi} = H_3 \tag{6.6}$$

where $H_3 = \frac{H^*}{A^*}$. Similarly, considering $A^* = B^* = 0$ and $C^* \neq 0$, one may also take

$$u_{\eta\eta} = H_4 \tag{6.7}$$

where $H_4 = \frac{H^*}{C^*}$ which is called the *second canonical form of the parabolic PDEs*.

Note that, for $B^2 - 4AC = 0$, the characteristic equations in (5.13) coincide. Thus, we obtain only a single integral $\xi =$ constant and η can be chosen freely to make the Jacobian (5.4) nonzero, for instance $\eta = y$ (or, without loss of generality, $\eta = x$). To see this, we consider

$$\begin{aligned} & B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0 \\ \overset{\eta = y}{\Longrightarrow} \quad & B^* = B\xi_x + 2C\xi_y = 0 \\ \overset{\substack{\xi = \text{const} \\ \eta = \text{const}}}{\Longrightarrow} \quad & \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{2C}{B} = \frac{4AC}{2AB} = \frac{B^2}{2AB} = \frac{B}{2A}. \end{aligned} \tag{6.8}$$

which are the *characteristic equations* for the parabolic PDEs.

Note also that, the same implication holds also when η is selected as $\eta = x$. The solution of this characteristic equation may be written as

$$\phi_1(x, y) = c_1, \text{ for constant } c_1. \tag{6.9}$$

Hence the transformations

$$\xi = \phi_1(x, y) \text{ and } \eta = y \tag{6.10}$$

will transform the PDE (5.10) into a canonical form.

Example 19

Find the general solution of the PDE

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0, \tag{6.11}$$

by obtaining its ***CANONICAL FORM***.

When $B^2 - 4AC < 0$, we have an elliptic PDE. After appropriate transformation, the determinant will be transformed to $B^{*2} - 4A^*C^* < 0$. For this case, we will consider the choice of $A^* = C^* \neq 0$ and $B^* = 0$ and this choice will result a to the following real canonical form:

$$u_{\xi\xi} + u_{\eta\eta} = H_5. \tag{6.18}$$

Here $H_5 = \frac{H^*}{A^*}$ and this equation is called the *real canonical form of the elliptic PDEs*. Moreover, the choice $A^* = C^* \neq 0$ and $B^* = 0$ yields

$$A^* - C^* = 0$$
$$\implies A(\xi_x^2 - \eta_x^2) + B(\xi_x\xi_y - \eta_x\eta_y) + C(\xi_y^2 - \eta_y^2) = 0, \tag{6.19a}$$
$$B^* = 0$$
$$\implies 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0 \tag{6.19b}$$

From (6.19a) and (6.19b), we obtain

$$\begin{aligned} &A^* - C^* + iB^* \\ &= A^*(\xi_x + i\eta_x)^2 + B^*(\xi_x + i\eta_x)(\xi_y + i\eta_y) + C^*(\xi_y + i\eta_y)^2 = 0 \\ &\Longrightarrow A^*\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B^*\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right) + C^* = 0 \end{aligned} \tag{6.20}$$

Note that, along the curves ξ =constant and η =constant, we have $d\xi = \xi_x dx + \xi_y dy = 0$ and $d\eta = \eta_x dx + \eta_y dy = 0$ which, in turn, imply $\frac{dy}{dx} = -\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}$. From this and the roots of (6.20), we obtain

$$\frac{dy}{dx} = \frac{B \pm i\sqrt{4AC - B^2}}{2A}. \tag{6.21}$$

The solutions of this complex characteristic equations may be obtained as

$$\Phi_1(x, y) = c_1 \text{ and } \Phi_2(x, y) = c_2, \text{ for } c_1,\ c_2 \text{ are constants.} \tag{6.22}$$

Defining $\Phi_1 := \xi + i\eta$ and $\Phi_2 := \xi - i\eta$, the following transformation is obtained

$$\begin{aligned} \xi =& \mathrm{Re}\,\Phi_1 = \frac{\Phi_1 + \Phi_2}{2}, \\ \eta =& \mathrm{Im}\,\Phi_2 = \frac{\Phi_1 - \Phi_2}{2i}. \end{aligned} \tag{6.23}$$

which will transform the PDE (5.10) into a real canonical form. Note that, the transformation (6.22) will transform the PDE (5.10) into the *complex canonical form of the elliptic PDEs* as

$$u_{\Phi_1 \Phi_2} = H_6 \tag{6.24}$$

where $H_6 = \frac{H^*}{iB^*}$.

Example 20

Find the general solution of the PDE

$$u_{xx} + x^2 u_{yy} = 0, \qquad (6.25)$$

by obtaining its ***(REAL/COMPLEX) CANONICAL FORM****.*