MTM3502-Partial Differential Equations

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Week 11

The Cauchy Problem: Homogeneous Wave Equations

As in first order PDEs, in order to find a particular solution of a given PDE (of second or higher order), there is a need for some *ICs* and/or *BCs* which is called the *Cauchy problem*. We, first, start with investigating Cauchy problems for hyperbolic PDEs. To study Cauchy problems for hyperbolic partial differential equations, it is quite natural to begin investigating the simplest and yet most important equation, the one-dimensional wave equation, by the method of characteristics.

Consider the following Cauchy problem of an infinite string with the IC

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{8.1a}$$

$$u(x,0) = f(x), \quad x \in \mathbb{R}, \tag{8.1b}$$

$$u_t(x,0) = g(x), \quad x \in \mathbb{R}. \tag{8.1c}$$

The Cauchy Problem: Homogeneous Wave Equations

By the method of characteristics, the characteristic equation will be

$$dx^2 - c^2 dt^2 = 0, \tag{8.2}$$

which reduces to

$$dx + cdt = 0, \quad dx - cdt = 0. \tag{8.3}$$

Integrating (8.3), we obtain the following characteristics

$$x + ct = c_1,\ x - ct = c_2 \implies \xi = x + ct,\ \eta = x - ct. \tag{8.4}$$

Evaluating the partial derivatives, we have

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}). \tag{8.5}$$

Substituting (8.5) into (8.1a) yields

$$-4c^2 u_{\xi\eta} = 0 \quad \stackrel{c \neq 0}{\Longrightarrow} \quad u_{\xi\eta} = 0 \tag{8.6}$$

Integrating with respect to η and ξ, we have

$$u(\xi,\eta) = \phi(\xi) + \psi(\eta) \overset{\substack{\xi = x + ct \\ \eta = x - ct}}{\Longrightarrow} \begin{aligned} u(x,y) =&\phi(x+ct) \\ &+ \psi(x-ct), \end{aligned} \tag{8.7}$$

where ϕ and ψ are (twice differentiable) arbitrary functions. This is called *the general solution of the wave equation*.

Now applying the initial conditions (8.1b) and (8.1c), we obtain

$$u(x,0) = f(x) = \phi(x) + \psi(x), \tag{8.8a}$$

$$u_t(x,0) = g(x) = c\phi'(x) - c\psi'(x). \tag{8.8b}$$

Integration of (8.8b) gives

$$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(\xi)d\xi + K \tag{8.9}$$

where x_0 and K are arbitrary constants.

Solving ϕ and ψ from (8.8a) and (8.9), we obtain

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\xi)d\xi + \frac{K}{2}, \tag{8.10a}$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)d\xi - \frac{K}{2}, \tag{8.10b}$$

and the solution is given as

$$\begin{aligned} u(x,t) =& \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \\ &+ \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(\xi)d\xi - \int_{x_0}^{x-ct} g(\xi)d\xi\right] \\ =& \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi \end{aligned} \tag{8.11}$$

This solution is called the well-known *d'Alembert solution* of the Cauchy problem for the one-dimensional wave equation.

Example 27

Find the solution of the IVP

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{8.12a}$$

$$u(x,0) = \sin x, \quad x \in \mathbb{R}, \tag{8.12b}$$

$$u_t(x,0) = \cos x, \quad x \in \mathbb{R}. \tag{8.12c}$$

The Cauchy Problem: Homogeneous Wave Equations

It follows from the d'Alembert solution that, if an initial displacement or an initial velocity is located in a small neighborhood of some point (x_0, t_0), it can influence only the area $t > t_0$ bounded by two characteristics $x - ct =$constant and $x + ct =$ constant with slope $\pm(1/c)$ passing through the point (x_0, t_0), as shown in Figure 1. This means that the initial displacement propagates with the speed $\frac{dx}{dt} = c$, whereas the effect of the initial velocity propagates at all speeds up to c. This infinite sector R in this figure is called *the range of influence* of the point (x_0, t_0).

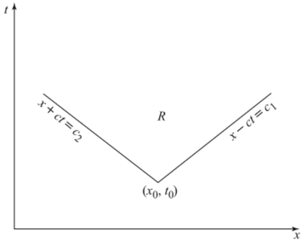

Figure: The Range of Influence of the Point (x_0, t_0).

The Cauchy Problem: Homogeneous Wave Equations

According to (8.11), the value of $u(x_0, t_0)$ depends on the initial data f and g in the interval $[x_0 - ct_0, x_0 + ct_0]$ which is cut out of the initial line by the two characteristics $x - ct =$constant and $x + ct =$constant with slope $\pm(1/c)$ passing through the point (x_0, t_0). The interval $[x_0 - ct_0, x_0 + ct_0]$ on the line $t = 0$ is called *the domain of dependence* of the solution at the point (x_0, t_0), as shown in Figure 3.

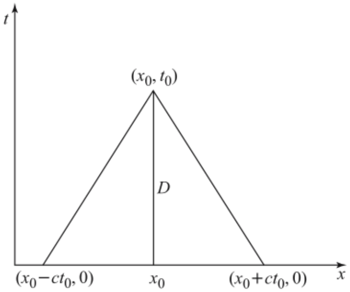

Figure: The Domain of Dependence of the Solution at the Point (x_0, t_0).

The Cauchy Problem: Homogeneous Wave Equations

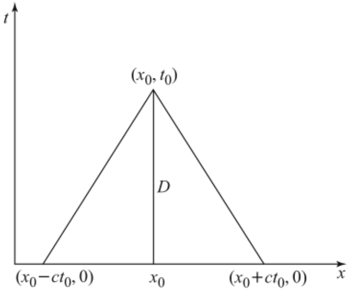

Figure: The Domain of Dependence of the Solution at the Point (x_0, t_0).

Since the solution $u(x, t)$ at every point (x, t) inside the triangular region D in this figure is completely determined by the Cauchy data on the interval $[x_0 - ct_0, x_0 + ct_0]$, the region D is called *the region of determinancy* of the solution.

The Cauchy Problem: Nonhomogeneous Wave Equations

Now, we consider the Cauchy problem for the nonhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = h^*(x,t), \quad x \in \mathbb{R}, \quad t > 0, \tag{8.14a}$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}, \tag{8.14b}$$
$$u_t(x,0) = g^*(x), \quad x \in \mathbb{R}. \tag{8.14c}$$

By coordinate transformation $y = ct$, the problem is reduced to

$$u_{xx} - u_{yy} = h(x,y), \quad x \in \mathbb{R}, \quad t > 0, \tag{8.15a}$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}, \tag{8.15b}$$
$$u_y(x,0) = g(x), \quad x \in \mathbb{R}. \tag{8.15c}$$

where $h \equiv -\frac{h^*}{c^2}$ and $g \equiv \frac{g^*}{c}$.

Let $P_0(x_0, y_0)$ be a point of the plane and let $Q(x_0, 0)$ be the point on the initial line $y = 0$. Then the characteristics, $x \pm y =$constant of (8.15a) are two straight lines drawn through the point P_0 with slopes ± 1. Obviously, they intersect the x-axis at the points $P_1(x_0 - y_0, 0)$ and $P_2(x_0 + y_0, 0)$, as shown in Figure 4.

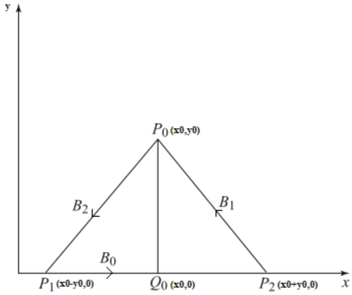

Figure: The Triangular Region R.

Let the sides of the triangle $P_0P_1P_2$ be designated by B_0, B_1 and B_2, and let R be the region representing the interior of the triangle and its boundaries B. Integrating both sides of equation (8.15a), we obtain

$$\int\int_R (u_{xx} - u_{yy})dxdy = \int\int_R h(x,y)dxdy. \qquad (8.16)$$

Now, by taking $M := u_x$ and $N := u_y$ (and, therefore, $u_{xx} = \frac{\partial N}{\partial x}$ and $u_{yy} = \frac{\partial M}{\partial y}$), we apply Green's theorem to obtain

$$\begin{aligned}\int\int_R (u_{xx} - u_{yy})dxdy &= \int\int_R \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) dxdy \\ &= \oint_B (Ndx + Mdy) \qquad (8.17) \\ &= \oint_B (u_y dx + u_x dy).\end{aligned}$$

Since $B = B_0 \cup B_1 \cup B_2$, we have three cases to consider:

- **On B_0:** Note that, since $x = x$ where x varies from $x = x_0 - y_0$ to $x = x_0 + y_0$ and $y = 0$ (and, hence, $dx = dx$ and $dy = 0$), we have

$$\begin{aligned} \int_{B_0} (u_x dy + u_y dx) &= \int_{B_0} u_y dx \\ &= \int_{x_0 - y_0}^{x_0 + y_0} u_y(x, 0) dx. \end{aligned} \tag{8.18}$$

Since $B = B_0 \cup B_1 \cup B_2$, we have three cases to consider:

- **On B_1:** Note that, since $y = -x + x_0 + y_0$ and x varies from $x = x_0 + y_0$ to $x = x_0$ (and, hence, $dy = -dx$), we have

$$\begin{aligned}\int_{B_1} (u_y dx + u_x dy) &= \int_{B_1} (u_x \cdot (-dx) + u_y \cdot (-dy)) \\ &= \int_{(x,y)=(x_0+y_0,0)}^{(x,y)=(x_0,y_0)} (-du) \\ &= u(x_0 + y_0, 0) - u(x_0, y_0).\end{aligned} \tag{8.19}$$

Since $B = B_0 \cup B_1 \cup B_2$, we have three cases to consider:

- **On B_2:** Note that, since $y = x - x_0 + y_0$ and x varies from $x = x_0$ to $x = x_0 - y_0$ (and, hence, $dy = dx$), we have

$$\begin{aligned} \int_{B_2} (u_y dx + u_x dy) &= \int_{B_2} (u_x dx + u_y dy) \\ &= \int_{(x,y)=(x_0,y_0)}^{(x,y)=(x_0-y_0,0)} du \\ &= u(x_0 - y_0, 0) - u(x_0, y_0). \end{aligned} \tag{8.20}$$

The Cauchy Problem: Nonhomogeneous Wave Equations

By (8.17), (8.18), (8.19) and (8.20), we have

$$\oint_B (u_y dx + u_x dy) = -2u(x_0, y_0) + u(x_0 - y_0, 0) + u(x_0 + y_0, 0) + \int_{x_0 - y_0}^{x_0 + y_0} u_y(x, 0) dx. \tag{8.21}$$

Combining (8.16), (8.17) and (8.21), we obtain

$$u(x_0, y_0) = \frac{1}{2}[u(x_0 - y_0, 0) + u(x_0 + y_0, 0)] + \frac{1}{2}\int_{x_0 - y_0}^{x_0 + y_0} u_y(x, 0) dx - \frac{1}{2}\int\int_R h(x, y) dx dy. \tag{8.22}$$

Since x_0 and y_0 are chosen arbitrarily, as a consequence, we replace x_0 by x and y_0 by y and (8.22) becomes

$$\begin{aligned} u(x,y) =& \frac{1}{2}[f(x-y)+f(x+y)] + \frac{1}{2}\int_{x-y}^{x+y} g(\xi)d\xi \\ & -\frac{1}{2}\int\int_R h(\xi,\eta)d\xi d\eta \end{aligned} \tag{8.23}$$

and replacing $y = ct$, we have

$$\begin{aligned} u(x,t) =& \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(\xi)d\xi \\ & +\frac{1}{2c}\int\int_R h^*(\xi,\tau)d\xi d\tau. \end{aligned} \tag{8.24}$$

The integration region R is a triangular region enclosed by $t = 0$ and characteristics so that it is named as *characteristic triangle*.

$$R = \{(\xi, \tau) \mid 0 \leq \tau \leq t, x - c(t - \tau) < \xi < x + c(t - \tau)\} \quad (8.25)$$

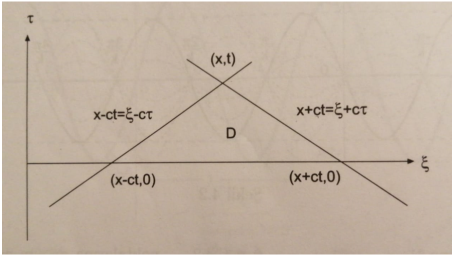

Figure: The Triangular Region R.

Therefore, (8.24) yields to

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(\xi)d\xi + \frac{1}{2c}\int_{\tau=0}^{\tau=t}\int_{\xi=x-ct+c\tau}^{\xi=x+ct-c\tau} h^*(\xi,\tau)d\xi d\tau. \tag{8.25}$$

Example 28

Determine the solution of

$$u_{tt} - c^2 u_{xx} = x, \tag{8.26a}$$

$$u(x,0) = \sin\ x, \tag{8.26b}$$

$$u_t(x,0) = x. \tag{8.26c}$$

Example 29

Find the solution of the IVP

$$u_{tt} - c^2 u_{xx} = 0, \quad t > 0, \tag{8.28a}$$

$$u(x, 0) = x^3, \tag{8.28b}$$

$$u_t(x, 0) = x, . \tag{8.28c}$$

Example 30

Find the solution of the IVP

$$u_{xx} - u_{tt} = 1, \tag{8.30a}$$

$$u(x,0) = \sin\ x, \tag{8.30b}$$

$$u_t(x,0) = x. \tag{8.30c}$$