MTM3502-Partial Differential Equations

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Week 13

The method of separation of variables is widely used to solve IVPs-BVPs involving linear PDEs. Usually, the dependent variable $u(x,y)$ is expressed in the separable form

$$u(x,y) = X(x)Y(y)$$

where X and Y are functions of x and y, respectively. Thereby, the PDE reduces to two ODEs for X and Y and the complete/general solution of the given PDE may be obtained thereafter.

Let us consider the second-order homogeneous partial differential equation

$$\begin{aligned} &A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} \\ &+ D(x,y)u_x + E(x,y)u_y + F(x,y)u = 0 \end{aligned} \tag{5.1-H}$$

where the coefficients A, B, C, D, E and F are the functions of x and y. We seek for a solution of the form $u(x,y) = X(x)Y(y)$; we, therefore, have the following partial derivatives

$$\begin{aligned} u_x &= X'(x)Y(y),\ u_y = X(x)Y'(y), \\ u_{xx} &= X''(x)Y(y),\ u_{xy} = X'(x)Y'(y), \\ u_{yy} &= X(x)Y''(y). \end{aligned} \tag{10.1}$$

Replacing these partial derivatives into (5.1-H), we have

$$\begin{aligned} AX''(x)Y(y) + BX'(x)Y'(y) + CX(x)Y''(y) \\ + DX'(x)Y(y) + EX(x)Y'(y) + FX(x)Y(y) = 0. \end{aligned} \tag{10.2}$$

The PDE (5.1-H) is considered as seperable if it can be written as

$$\frac{1}{X} f(x, D_x)X = \frac{1}{Y} g(y, D_y)Y \tag{10.3}$$

where f and g are quadratic functions of D_x and D_y, respectively.

The Method of Separation of Variables: The Complete Solution

To this regard, this is only possible when two functions of two different independent variables are constant. Thus, we have

$$\frac{1}{X}f(x, D_x)X = \frac{1}{Y}g(y, D_y)Y = \lambda \tag{10.4}$$

where λ is called the separation constant. Hence, we obtain two second order ODEs

$$f(x, D_x)X - \lambda X = 0, \quad g(y, D_y)Y - \lambda Y = 0 \tag{10.5}$$

and we have ODE techniques to deal with them.

Example 31

Find the solution of the PDE

$$u_{xx} + y^2 u_{yy} + y u_y = 0 \tag{10.6}$$

by using the method of separation of variables.

TMSV: The Particular Solution by Fourier Series Expansion

Let us seek for a solution of the PDE

$$u_{xx} + y^2 u_{yy} + y u_y = 0 \tag{10.6}$$

satisfying the following BVs

$$\begin{aligned} u(0, y) &= 0, \quad u(x, 0) = 0, \\ u(2\pi, y) &= 0, \quad u(x, 1) = f(x), \end{aligned} \tag{10.19}$$

on $D = \{(x, y) \mid 0 \leq x \leq 2\pi,\ 0 \leq y \leq 1\}$. Suppose that f is periodic, i.e. $f(0) = f(2\pi)$.

TMSV: The Particular Solution by Fourier Series Expansion

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The following theorem gives us the general uniform convergence condition of $c_0 + \sum_{n=1}^{\infty}(c_{1n}\cos(nx) + c_{2n}\sin(nx))$ to a periodic function f.

Theorem

Let f be a periodic function with $f(0) = f(2\pi)$ defined on $[0, 2\pi]$. If f is piecewise continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$, then the series

$$S(x) = c_0 + \sum_{n=1}^{\infty}(c_{1n}\cos(nx) + c_{2n}\sin(nx)) \tag{10.33}$$

with

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx, \\ c_{1n} &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(nx)dx, \\ c_{2n} &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin(nx)dx, \end{aligned} \tag{10.34}$$

converges to $f(x)$ at continuous points of f and converges to $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$ at discontinuous points of f.