MTM3502-Partial Differential Equations

Gökhan Göksu, PhD

Week 2

Solutions of FO Linear, Semi-/Quasi-Linear PDEs: Lagrange's Method

Last week, we introduced the structural classification of PDEs according to their linearity. By definition, a linear PDE is also a semi-linear PDE, a semi-linear PDE is a quasi-linear PDE and the nonlinear PDEs constitutes the general class for first order PDEs. However, the converse implication does not hold. See the following relations among the classes of first order PDEs.

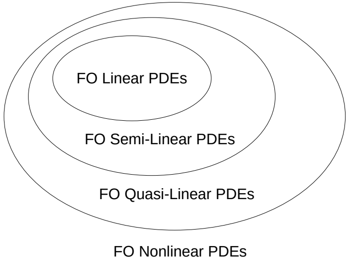

Figure: The Relations among the Classes of First Order PDEs.

Solutions of FO Linear, Semi-/Quasi-Linear PDEs: Lagrange's Method

To this regard, we, first, present the Lagrange's method for first order quasi-linear PDEs which is also valid for first order linear and semi-linear PDEs.

Theorem

Recall the quasi-linear PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{1.12}$$

The general solution of (1.12) *is of the form*

$$\varphi(\eta,\xi) = 0 \tag{2.1}$$

where φ *is an arbitrary function of*

$$\begin{aligned} \eta(x,y,u) &= c_1, \\ \xi(x,y,u) &= c_2, \\ c_1, c_2 &\in \mathbb{R}, \end{aligned} \tag{2.2}$$

which form a solution of

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}. \tag{2.3}$$

Proof.

Since $\eta(x, y, u) = c_1$ is a solution of (2.3), the total derivative of η

$$\frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy + \frac{\partial \eta}{\partial u}du = 0 \tag{2.4}$$

and (2.3) must be compatible. Therefore, we have

$$a(x, y, u)\eta_x + b(x, y, u)\eta_y + c(x, y, u)\eta_u = 0. \tag{2.5}$$

Similarly, we can also obtain

$$a(x, y, u)\xi_x + b(x, y, u)\xi_y + c(x, y, u)\xi_u = 0. \tag{2.6}$$

Solving (2.5) and (2.6) for the functions a, b and c, we have

$$\frac{a(x, y, u)}{\eta_y \xi_u - \eta_u \xi_y} = \frac{b(x, y, u)}{\eta_u \xi_x - \eta_x \xi_u} = \frac{c(x, y, u)}{\eta_x \xi_y - \eta_y \xi_x}. \tag{2.7}$$

Proof (Continued).

Considering (2.1) and differentiating it partially with respect to x and y yields

$$\frac{\partial\varphi}{\partial\eta}\left(\frac{\partial\eta}{\partial x}+\frac{\partial\eta}{\partial u}\frac{\partial u}{\partial x}\right)+\frac{\partial\varphi}{\partial\xi}\left(\frac{\partial\xi}{\partial x}+\frac{\partial\xi}{\partial u}\frac{\partial u}{\partial x}\right)=\varphi_\eta\left(\eta_x+p\eta_u\right)+\varphi_\xi\left(\xi_x+p\xi_u\right)=0, \tag{2.8a}$$

$$\frac{\partial\varphi}{\partial\eta}\left(\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial u}\frac{\partial u}{\partial y}\right)+\frac{\partial\varphi}{\partial\xi}\left(\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial u}\frac{\partial u}{\partial y}\right)=\varphi_\eta\left(\eta_y+q\eta_u\right)+\varphi_\xi\left(\xi_y+q\xi_u\right)=0. \tag{2.8b}$$

Writing (2.8a) and (2.8b) as a system yields to

$$\begin{bmatrix}\eta_x+p\eta_u & \xi_x+p\xi_u\\ \eta_y+q\eta_u & \xi_y+q\xi_u\end{bmatrix}\begin{bmatrix}\varphi_\eta\\ \varphi_\xi\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{2.9}$$

Proof (Continued).

and a nontrivial solution for $\begin{bmatrix} \varphi_\eta & \varphi_\xi \end{bmatrix}^\top$ can be obtained only in the following case:

$$\begin{vmatrix} \eta_x + p\eta_u & \xi_x + p\xi_u \\ \eta_y + q\eta_u & \xi_y + q\xi_u \end{vmatrix} = 0$$
$$\implies \left(\eta_x + p\eta_u\right)\left(\xi_y + q\xi_u\right) - \left(\xi_x + p\xi_u\right)\left(\eta_y + q\eta_u\right) = 0. \tag{2.10}$$

Arranging the terms in (2.10), we obtain

$$p\left(\eta_y\xi_u - \eta_u\xi_y\right) + q\left(\eta_u\xi_x - \eta_x\xi_u\right) = \left(\eta_x\xi_y - \eta_y\xi_x\right). \tag{2.11}$$

From equations (2.7) and (2.11) implies

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{1.12}$$

which concludes the proof. □

A similar result can also be shown for first order linear and semi-linear PDEs.

Corollary

Recall the linear PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \tag{1.8}$$

The general solution of (1.8) *is of the form* φ *of* (2.1) *where* φ *is an arbitrary function of* η *and* ξ *of* (2.2) *which form a solution of*

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{d(x,y) - c(x,y)u}. \tag{2.12}$$

Corollary

Recall the semi-linear PDE

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \tag{1.10}$$

The general solution of (1.10) *is of the form* φ *of* (2.1) *where* φ *is an arbitrary function of* η *and* ξ *of* (2.2) *which form a solution of*

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)}. \tag{2.13}$$

Remark

Equating (2.3), (2.12) *and* (2.13) *with* dt, *it is possible to obtain the characteristic equations of first order quasi-linear, linear and semi-linear PDEs, which were presented in (1.13), (1.9) and (1.11), respectively. We will investigate it later in detail.*

Example 4

Find the general solution of the first order linear PDE

$$u(xp - yq) = y^2 - x^2 \tag{2.14}$$

Example 5

Find the general solution of the PDE

$$x(x+y)p = y(x+y)q - (x-y)(2x+2y+u) \qquad (2.19)$$

Integral Surfaces Passing Through a Given Curve

Recall the quasi-linear PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{1.12}$$

and let the parametric equations of the given curve be

$$x = x(t),\ y = y(t),\ u = u(t),\ t \geq 0. \tag{2.33}$$

Let also

$$\eta(x,y,u) = c_1,\ \xi(x,y,u) = c_2 \tag{2.34}$$

be any two solutions of the systems of the equations

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}. \tag{2.3}$$

Then the general solution of (1.12) is

$$\varphi(\eta, \xi) = 0 \tag{2.1}$$

where φ is an arbitrary function. Since the integral surface has to pass through (2.33), we obtain

$$\eta(x(t), y(t), u(t)) = c_1,\ \xi(x(t), y(t), u(t)) = c_2 \tag{2.35}$$

subject to the condition that

$$\varphi(c_1, c_2) = 0 \tag{2.36}$$

Hence, the required integral surface can be obtained by eliminating c_1 and c_2 from (2.34), (2.3) and (2.36).

Example 6

Let us find the equation of the integral surface of the PDE

$$2y(u-3)p+(2x-u)q=y(2x-3) \tag{2.37}$$

which passes through the circle $z=0$, $x^2+y^2=2x$.

Example 7

Find the general integral of the PDE

$$(x-y)p+(y-x-u)q=u \tag{2.45}$$

and the particular solution through the circle

$$u=1,\ x^2+y^2=1. \tag{2.46}$$

Remark

In order to find the integral surface passing through a given curve;

- *The procedure generally starts by finding the parametrization of the curve.*
- *Then, a relation between the constants is obtained from parametrization.*
- *Latter, the particular solution can be obtained from the constants obtained from the solution of the PDE.*