MTM3502-Partial Differential Equations

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Week 3

Let us consider the system of surfaces given by

$$\phi(x, y, u) = c, \;\; c \in \mathbb{R}. \tag{3.1}$$

and let the required surface orthogonal to the system of surfaces (3.1) be

$$u = u(x, y). \tag{3.2}$$

Consider the point of the curve of the intersection of (3.1) and (3.2) and denote it as $P(x, y, u)$. The normal vectors of the both surfaces at $P(x, y, u)$ are given by the gradient vectors as

$$\nabla\phi = \begin{bmatrix} \phi_x & \phi_y & \phi_u \end{bmatrix}, \; \nabla u = \begin{bmatrix} u_x & u_y & -1 \end{bmatrix}. \tag{3.3}$$

Since the surfaces are orthogonal, the normal vectors at $P(x, y, u)$ will be orthogonal. Thus, the inner product of $\nabla\phi$ and ∇u should be 0:

$$\langle\nabla\phi, \nabla u\rangle = \phi_x u_x + \phi_y u_y - \phi_u = 0 \tag{3.4}$$

which is of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{1.12}$$

where $a(x, y, u) = \phi_x$, $b(x, y, u) = \phi_y$ and $c(x, y, u) = \phi_u$. Then, any surface passing through the integral curves given by (1.12) is the required surface, which will be orthogonal to (3.1).

Example 8

Find the surface which is orthogonal to the one parameter system

$$u = cxy(x^2 + y^2),\ c \in \mathbb{R} \setminus \{0\} \tag{3.5}$$

and which passes through the hyperbola

$$x^2 - y^2 = a^2,\ u = 0,\ a > 0. \tag{3.6}$$

Compatibility of First Order PDEs

Consider the two first order nonlinear PDEs

$$\begin{aligned} F(x, y, u, p, q) &= 0, \qquad (3.16a)\\ G(x, y, u, p, q) &= 0. \qquad (3.16b)\end{aligned}$$

The compatibility condition for the solutions of (3.16a) and (3.16b) is given as the following.

Theorem (Condition for Compatibility)

Consider the PDEs (3.16a) *and* (3.16b) *and assume that*

$$J = \frac{\partial(F, G)}{\partial(p, q)} = \begin{vmatrix} F_p & F_q \\ G_p & G_q \end{vmatrix} \neq 0. \qquad (3.17)$$

The PDEs (3.16a) *and* (3.16b) *have a common solution, i.e. "compatible", if and only if the following holds*

$$[F, G] = 0 \qquad (3.18)$$

where

$$[F, G] = \frac{\partial(F, G)}{\partial(x, p)} + p\frac{\partial(F, G)}{\partial(u, p)} + \frac{\partial(F, G)}{\partial(y, q)} + q\frac{\partial(F, G)}{\partial(u, q)}. \qquad (3.19)$$

Proof (Sketch).

The implication holds by rewriting (3.16a) and (3.16b) as $p = p(x, y, u)$ and $q = q(x, y, u)$ and noticing that $pdx + qdy - du = 0$ is integrable if and only if

$$\begin{aligned}
\begin{bmatrix} p & q & -1 \end{bmatrix} \cdot \nabla \times \begin{bmatrix} p \\ q \\ -1 \end{bmatrix} &= \begin{bmatrix} p & q & -1 \end{bmatrix} \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial u} \\ p & q & -1 \end{vmatrix} \\
&= \begin{bmatrix} p & q & -1 \end{bmatrix} \begin{bmatrix} -q_u \\ -p_u \\ q_x - p_y \end{bmatrix} \qquad (3.20) \\
&= -pq_u - qp_u + p_y - q_x = 0
\end{aligned}$$

holds. □

Example 9

Show that the equations

$$xp - yq = x, \; x^2p + q = xz \tag{3.21}$$

are compatible and find their solution.

A solution of a PDE is a relation between the dependent variable (in our case u) and independent variables (in our case x and y) which satisfies the PDE. So far, we have demonstrated the solutions of first order PDEs with an arbitrary function. However this is not the only way to represent the solutions of PDEs.

Previous week, we also showed that the system of surfaces can be formulated as a PDE of first order. We differentiated the system of surfaces partially with respect to independent variables to verify that these system of surfaces are the solutions of the given PDEs.

Now, we will see how we can verify that the given solution, in form of a system of surfaces, is a solution of the corresponding PDE and we will classify the types of these solutions. To that aim, let us first consider the general class of first order (nonlinear) PDE

$$F(x, y, u, p, q) = 0. \tag{1.17}$$

Let us assume that there exists a (parameter-varying) relation

$$\phi(x, y, u) = \tilde{\phi}(x, y, u; a, b) = 0, \tag{1.18}$$

such that the arbitrary constants a and b can be eliminated from (1.18) and (1.17) is satisfied by considering the partial derivatives p and q. Then, (1.18) is called an integral (solution) of (1.17), where a and b are arbitrary constants.

Classification of the SOLUTIONS of First Order PDEs

Let us now define the types of integrals of a PDE. There are four types of integrals of a PDE:

- **Complete Integral:** The integral of (1.17) represented by (1.18) where a and b are arbitrary constants, is called a complete integral and it represents two paramater system of surfaces. In general case, for complete integral of a PDE, the number of arbitrary constants should coincide with the number of independent variables. For example, $u = ax + by$ is a complete solution of $u = px + qy$.
- **General Integral:** If the parameter b in (1.18) is connected with a through $b = \phi(a)$, where ϕ is an arbitrary function, then the envelope of the resulting equation will be a solution of (1.17) and is called the general integral. In general case, for general integral of a PDE, the number of arbitrary functions should coincide with the order of the PDE. For example, $u = x\phi(\frac{y}{x})$, where ϕ is an arbitrary function, is the general solution of $u = px + qy$.

- **Particular Integral:** When a particular form of the arbitrary function ϕ is known, the corresponding integral is called a particular integral of (1.17). For example, $u = 4x - 3y$ or $u = x\sin(\frac{y}{x})$ are particular solutions of $u = px + qy$.
- **Singular Integral (*Tur: Aykırı Çözüm*):** If the envelope of the two parameter system of surfaces (1.18) exists, it is also a solution of (1.17) and is called the singular integral of the given PDE. To obtain the singular solution, we start by differentiating the complete solution with respect to the arbitrary constants. The elimination of the constants from these equations (if possible) is the singular solution. We will now see how to obtain singular integrals of first order PDEs in detail.

Example 10

Find the complete integral of

$$p + q = 4 \tag{3.27}$$

and the singular integral of (3.27)*, if exists.*

Example 11

Find the complete integral of

$$pq + 2 = 0 \tag{3.33}$$

and the singular integral of (3.33)*, if exists.*

Example 12

Now, consider the following PDE

$$u = px + qy + \varphi(p, q) \tag{3.37}$$

where φ is an arbitrary function.

Example 13

Find the singular integral of the PDE

$$u^2(1+p^2+q^2)=1 \tag{3.44}$$

where the complete integral of (3.44) *is given by*

$$\tilde{\phi}(x,y,u;a,b)=(x-a)^2+(y-b)^2+u^2-1=0. \tag{3.45}$$