MTM3502-Partial Differential Equations

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Week 7

The general solutions of hyperbolic PDEs are obtained by their canonical forms. However, sometimes, it might not be so simple to determine the general solution of a given equation. If the canonical form of the equation is simple, then the general solution can be immediately ascertained. Let us recall the example of last week.

Example 18

Consider the PDE

$$y^2 u_{xx} - x^2 u_{yy} = 0. \tag{5.18}$$

The canonical form of the given equation was

$$u_{\xi\eta} = \frac{\eta}{2(\xi^2 - \eta^2)} u_\xi - \frac{\xi}{2(\xi^2 - \eta^2)} u_\eta. \tag{5.21}$$

where $\xi = y^2 - x^2$ and $\eta = y^2 + x^2$.

Multiplying both sides with -4, we obtain

$$\begin{aligned} -4u_{\xi\eta} &= \frac{2\eta}{\eta^2-\xi^2}u_\xi - \frac{2\xi}{\eta^2-\xi^2}u_\eta \\ \implies \quad -4u_{\xi\eta} &= \Big(\ln(\eta^2-\xi^2)\Big)_\eta u_\xi + \Big(\ln(\eta^2-\xi^2)\Big)_\xi u_\eta. \end{aligned} \tag{6.1}$$

Without loss of generality, let us choose

$$-2u_{\xi\eta} = \Big(\ln(\eta^2-\xi^2)\Big)_\eta u_\xi, \quad (u_1 = u_\xi), \tag{6.2a}$$

$$-2u_{\xi\eta} = \Big(\ln(\eta^2-\xi^2)\Big)_\xi u_\eta, \quad (u_2 = u_\eta), \tag{6.2b}$$

From (6.2a), we have

$$
\begin{aligned}
\frac{du_1}{u_1} &= -\frac{\left(\ln(\eta^2-\xi^2)\right)_\eta}{2}\,d\eta \\
\implies \quad \ln u_1 &= -\frac{\ln(\eta^2-\xi^2)}{2} + \ln\varphi(\xi) \\
\implies \quad u_1 = u_\eta &= \frac{\varphi(\xi)}{\sqrt{\eta^2-\xi^2}} \\
\implies \quad u &= \varphi(\xi)\tanh^{-1}\left(\frac{\eta}{\sqrt{\eta^2-\xi^2}}\right) + \varphi_1(\xi)
\end{aligned}
\tag{6.3}
$$

Similarly, from (6.2b), we have

$$
\begin{aligned}
u_2 = u_\xi &= \frac{\phi(\eta)}{\sqrt{\eta^2-\xi^2}} \\
\implies \quad u &= \phi(\eta)\tan^{-1}\left(\frac{\xi}{\sqrt{\eta^2-\xi^2}}\right) + \phi_1(\eta)
\end{aligned}
\tag{6.4}
$$

Solving (6.3) and (6.4) together, we obtain

$$u(\xi,\eta) = c_1 \tanh^{-1}\left(\frac{\eta}{\sqrt{\eta^2-\xi^2}}\right)\tan^{-1}\left(\frac{\xi}{\sqrt{\eta^2-\xi^2}}\right) + c_2$$
$$\implies \quad u(x,y) = c_1 \tanh^{-1}\left(\frac{y^2+x^2}{2xy}\right)\tan^{-1}\left(\frac{y^2-x^2}{2xy}\right) + c_2 \tag{6.5}$$

which is a solution to (5.18) where c_1 and c_2 are arbitrary constants.

The general form of a second order linear PDE was given as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G. \tag{5.1}$$

If A, B, C, D, E and F are real constants and G is a continuous function of x and y, then (5.1) is called a second order linear PDE with constant coefficients. This week, we will consider PDEs of these particular type and present an operator factorization technique for these type of PDEs.

In order to restate (5.1) in operator form, we define the following partial differential operators

$$\begin{gathered} D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}, \\ D_x^2 = \frac{\partial^2}{\partial x^2}, D_y^2 = \frac{\partial^2}{\partial y^2}, D_x D_y = \frac{\partial^2}{\partial x \partial y}. \end{gathered} \tag{7.1}$$

Then, (5.1) will be restated as

$$Lu = G(x, y) \tag{7.2}$$

where

$$L = AD_x^2 + BD_xD_y + CD_y^2 + DD_x + ED_y + F \qquad (7.3)$$

is the partial differential operator for (5.1). The operator L is a linear operator which satisfy the following property, for real constants c_1 and c_2 and partially second order differentiable functions u_1 and u_2,

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2. \qquad (7.3)$$

Suppose that u_1 and u_2 are the solutions to the homogeneous PDE

$$Lu = 0 \implies Lu_1 = 0 \text{ and } Lu_2 = 0. \tag{7.4}$$

Then, from linearity of L, the linear combination of u_1 and u_2, which is $c_1 u_1 + c_2 u_2$ for nonzero reals c_1 and c_2, is also a solution of (7.4). This property is called as the *superposition principle*. The general solution of (7.4) is called the *homogeneous solution* and denoted as u_h whereas the particular solution to (7.2) is called the *particular solution* and denoted as u_p. Thus, the general solution of (7.2) have the following form

$$u = u_h + u_p. \tag{7.5}$$

Suppose that the operator L can be factorized as

$$L = L_1 L_2 = (\alpha_1 D_x + \beta_1 D_y + \gamma_1)(\alpha_2 D_x + \beta_2 D_y + \gamma_2) \tag{7.6}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, for $i = 1, 2$. If $u = u(x, t)$ is a solution of $L_2 u = 0$, then $Lu = L_1 L_2 u = L_1 \cdot 0 = 0$ so that it is also a solution for $Lu = 0$ (the same can also shown for $L_1 u = 0$, without loss of generality). Now, let us consider

$$L_1 u = (\alpha_1 D_x + \beta_1 D_y + \gamma_1)u = \alpha_1 p + \beta_1 q + \gamma_1 u = 0. \tag{7.7}$$

From the Lagrange auxiliary equation, we have

$$\frac{dx}{\alpha_1} = \frac{dy}{\beta_1} = \frac{du}{-\gamma_1 u}. \tag{7.8}$$

Now, we consider three cases:

- Suppose $\alpha_1 \neq 0$ holds. From (7.8), we have

$$\begin{aligned} \frac{dx}{\alpha_1} = \frac{dy}{\beta_1} \quad &\Longrightarrow \quad \beta_1 x - \alpha_1 y = c_1 \\ -\frac{\gamma_1}{\alpha_1} dx = \frac{du}{u} \quad &\Longrightarrow \quad u = c_2 e^{-\frac{\gamma_1}{\alpha_1} x}. \end{aligned} \tag{7.9}$$

  Therefore, the general solution of (7.7) will be

$$u_{h1} = e^{-\frac{\gamma_1}{\alpha_1} x} \varphi(\beta_1 x - \alpha_1 y), \tag{7.10}$$

  where φ is an arbitrary function.

From the Lagrange auxiliary equation, we have

$$\frac{dx}{\alpha_1} = \frac{dy}{\beta_1} = \frac{du}{-\gamma_1 u}. \tag{7.8}$$

Now, we consider three cases:

- Suppose $\beta_1 \neq 0$ holds. From (7.8), the general solution of (7.7) will be

$$u_{h1} = e^{-\frac{\gamma_1}{\beta_1} y} \tilde{\varphi}(\beta_1 x - \alpha_1 y), \tag{7.11}$$

  where $\tilde{\varphi}$ is an arbitrary function.
- In the general case that both $\alpha_1 \neq 0$ and $\beta_1 \neq 0$ hold, either (7.10) or (7.11) may be chosen for general solution.

Similarly, the general solution of

$$\begin{aligned} L_2 u =& (\alpha_2 D_x + \beta_2 D_y + \gamma_2) u \\ =& \alpha_2 p + \beta_2 q + \gamma_2 u = 0 \end{aligned} \tag{7.12}$$

will be either

$$u_{h2} = e^{-\frac{\gamma_2}{\alpha_2} x} \phi(\beta_2 x - \alpha_2 y), \tag{7.13}$$

for $\alpha_2 \neq 0$ or

$$u_{h2} = e^{-\frac{\gamma_2}{\beta_2} y} \tilde{\phi}(\beta_2 x - \alpha_2 y), \tag{7.14}$$

for $\beta_2 \neq 0$. As a result, the general homogeneous solution of $Lu = 0$ will be

$$\begin{aligned} u_h =& u_{h1} + u_{h2} \\ =& e^{-\frac{\gamma_1}{\alpha_1} x} \varphi(\beta_1 x - \alpha_1 y) + e^{-\frac{\gamma_2}{\alpha_2} x} \phi(\beta_2 x - \alpha_2 y). \end{aligned} \tag{7.15}$$

In the case that $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, the homogeneous solutions of $Lu = 0$ will be

$$u_h = e^{-\frac{\gamma}{\alpha}x}\left(x\varphi(\beta x - \alpha y) + \phi(\beta x - \alpha y)\right). \tag{7.16}$$

for $\alpha \neq 0$ and

$$u_h = e^{-\frac{\gamma}{\beta}y}\left(y\varphi(\beta x - \alpha y) + \phi(\beta x - \alpha y)\right). \tag{7.17}$$

for $\beta \neq 0$.

The particular solution requires that we make an “initial assumption" about the form of the particular solution $u_p(x, y)$, but with the coefficients left unspecified. For the particular solution, the type of $G(x, y)$ in (7.2) determines the types of the particular solution candidate. Here are some of the following particular solution candidates for various types.

Table: Particular Solution Candidates for Various Types of $G(x, y)$.

$G(x, y)$	Particular Solution Candidate $u_p(x, y)$
Trigonometric	Trigonometric
Polynomial	Polynomial
Exponential	Exponential

Example 21

Find the homogeneous and particular solution of

$$u_{xx} - u_{yy} - u_x + u_y = 2\cos(3x + 2y). \tag{7.18}$$

Example 22

Find the general solution of

$$4u_{xx} - 4u_{xy} + u_{yy} + 4u_x - 2u_y = 0. \qquad (7.29)$$

Example 23

Find the general solution of

$$(2D_x^2 - 5D_xD_y + 2D_y^2)u = 5e^{x+3y}. \qquad (7.32)$$

where D_x^2, D_xD_y *and* D_y^2 *are defined as in* (7.1)