# MTM3502-Partial Differential Equations

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Week 13

In parabolic PDEs, since $B^2 - 4AC = 0$, the determinant yields to ${B^*}^2 - 4A^*C^* = 0$. We have two cases to obtain canonical forms of hyperbolic PDEs:

- $A^* = B^* = 0$ and $C^* \neq 0$,
- $A^* \neq 0$ and $B^* = C^* = 0$.

The *first canonical form of the parabolic PDEs*, considering $A^* \neq 0$ and $B^* = C^* = 0$, is

$$u_{\xi\xi} = H_3 \tag{6.6}$$

where $H_3 = \frac{H^*}{A^*}$. Similarly, considering $A^* = B^* = 0$ and $C^* \neq 0$, one may also take

$$u_{\eta\eta} = H_4 \tag{6.7}$$

where $H_4 = \frac{H^*}{C^*}$ which is called the *second canonical form of the parabolic PDEs*.

Note that, for $B^2 - 4AC = 0$, the characteristic equations in (5.13) coincide. Thus, we obtain only a single integral $\xi =$ constant and $\eta$ can be chosen freely to make the Jacobian (5.4) nonzero, for instance $\eta = y$ (or, without loss of generality, $\eta = x$). To see this, we consider

$$\begin{aligned} & B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0 \\ & \overset{\eta = y}{\Longrightarrow} \quad B^* = B\xi_x + 2C\xi_y = 0 \\ & \overset{\substack{\xi = \text{const} \\ \eta = \text{const}}}{\Longrightarrow} \quad \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{2C}{B} = \frac{4AC}{2AB} = \frac{B^2}{2AB} = \frac{B}{2A}. \end{aligned} \tag{6.8}$$

which are the *characteristic equations* for the parabolic PDEs.

Note also that, the same implication holds also when $\eta$ is selected as $\eta = x$. The solution of this characteristic equation may be written as

$$\phi_1(x, y) = c_1, \text{ for constant } c_1. \tag{6.9}$$

Hence the transformations

$$\xi = \phi_1(x, y) \text{ and } \eta = y \tag{6.10}$$

will transform the PDE (5.10) into a canonical form.

## Example 19

*Find the general solution of the PDE*

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0, \tag{6.11}$$

*by obtaining its **CANONICAL FORM**.*

When $B^2 - 4AC < 0$, we have an elliptic PDE. After appropriate transformation, the determinant will be transformed to $B^{*2} - 4A^*C^* < 0$. For this case, we will consider the choice of $A^* = C^* \neq 0$ and $B^* = 0$ and this choice will result a to the following real canonical form:

$$u_{\xi\xi} + u_{\eta\eta} = H_5. \tag{6.18}$$

Here $H_5 = \frac{H^*}{A^*}$ and this equation is called the *real canonical form of the elliptic PDEs*. Moreover, the choice $A^* = C^* \neq 0$ and $B^* = 0$ yields

$$\begin{aligned}
&A^* - C^* = 0 \\
&\quad \Longrightarrow \quad A(\xi_x^2 - \eta_x^2) + B(\xi_x\xi_y - \eta_x\eta_y) + C(\xi_y^2 - \eta_y^2) = 0, \quad (6.19a) \\
&B^* = 0 \\
&\quad \Longrightarrow \quad 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0 \quad (6.19b)
\end{aligned}$$

From (6.19a) and (6.19b), we obtain

$$\begin{aligned} &A^* - C^* + iB^* \\ &= A^*(\xi_x + i\eta_x)^2 + B^*(\xi_x + i\eta_x)(\xi_y + i\eta_y) + C^*(\xi_y + i\eta_y)^2 = 0 \\ &\Longrightarrow A^*\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B^*\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right) + C^* = 0 \end{aligned} \tag{6.20}$$

Note that, along the curves $\xi$ =constant and $\eta$ =constant, we have $d\xi = \xi_x dx + \xi_y dy = 0$ and $d\eta = \eta_x dx + \eta_y dy = 0$ which, in turn, imply $\frac{dy}{dx} = -\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}$. From this and the roots of (6.20), we obtain

$$\frac{dy}{dx} = \frac{B \pm i\sqrt{4AC - B^2}}{2A}. \tag{6.21}$$

The solutions of this complex characteristic equations may be obtained as

$$\Phi_1(x, y) = c_1 \text{ and } \Phi_2(x, y) = c_2, \text{ for } c_1,\ c_2 \text{ are constants.} \tag{6.22}$$

Defining $\Phi_1 := \xi + i\eta$ and $\Phi_2 := \xi - i\eta$, the following transformation is obtained

$$\begin{aligned} \xi &= \mathrm{Re}\,\Phi_1 = \frac{\Phi_1 + \Phi_2}{2}, \\ \eta &= \mathrm{Im}\,\Phi_2 = \frac{\Phi_1 - \Phi_2}{2i}. \end{aligned} \tag{6.23}$$

which will transform the PDE (5.10) into a real canonical form. Note that, the transformation (6.22) will transform the PDE (5.10) into the *complex canonical form of the elliptic PDEs* as

$$u_{\Phi_1 \Phi_2} = H_6 \tag{6.24}$$

where $H_6 = \frac{H^*}{iB^*}$.

## Example 20

*Find the general solution of the PDE*

$$u_{xx} + x^2 u_{yy} = 0, \tag{6.25}$$

*by obtaining its* ***(REAL/COMPLEX) CANONICAL FORM****.*

As in first order PDEs, in order to find a particular solution of a given PDE (of second or higher order), there is a need for some *ICs* and/or *BCs* which is called the *Cauchy problem*. We, first, start with investigating Cauchy problems for hyperbolic PDEs. To study Cauchy problems for hyperbolic partial differential equations, it is quite natural to begin investigating the simplest and yet most important equation, the one-dimensional wave equation, by the method of characteristics.

Consider the following Cauchy problem of an infinite string with the IC

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{8.1a}$$

$$u(x,0) = f(x), \quad x \in \mathbb{R}, \tag{8.1b}$$

$$u_t(x,0) = g(x), \quad x \in \mathbb{R}. \tag{8.1c}$$

## D'Alembert's Solution for the Homogeneous Wave Equation

By the method of characteristics, the characteristic equation will be

$$dx^2 - c^2 dt^2 = 0, \tag{8.2}$$

which reduces to

$$dx + cdt = 0, \quad dx - cdt = 0. \tag{8.3}$$

Integrating (8.3), we obtain the following characteristics

$$x + ct = c_1,\ x - ct = c_2 \implies \xi = x + ct,\ \eta = x - ct. \tag{8.4}$$

Evaluating the partial derivatives, we have

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}). \tag{8.5}$$

Substituting (8.5) into (8.1a) yields

$$-4c^2 u_{\xi\eta} = 0 \quad \overset{c \neq 0}{\Longrightarrow} \quad u_{\xi\eta} = 0 \tag{8.6}$$

Integrating with respect to $\eta$ and $\xi$, we have

$$u(\xi,\eta)=\phi(\xi)+\psi(\eta)\quad\overset{\substack{\xi=x+ct\\ \eta=x-ct}}{\Longrightarrow}\quad u(x,y)=\phi(x+ct)+\psi(x-ct), \tag{8.7}$$

where $\phi$ and $\psi$ are (twice differentiable) arbitrary functions. This is called *the general solution of the wave equation*.

Now applying the initial conditions (8.1b) and (8.1c), we obtain

$$u(x,0)=f(x)=\phi(x)+\psi(x), \tag{8.8a}$$

$$u_t(x,0)=g(x)=c\phi'(x)-c\psi'(x). \tag{8.8b}$$

Integration of (8.8b) gives

$$\phi(x)-\psi(x)=\frac{1}{c}\int_{x_0}^{x}g(\xi)d\xi+K \tag{8.9}$$

where $x_0$ and $K$ are arbitrary constants.

# D'Alembert's Solution for the Homogeneous Wave Equation

Solving $\phi$ and $\psi$ from (8.8a) and (8.9), we obtain

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\xi)d\xi + \frac{K}{2}, \tag{8.10a}$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)d\xi - \frac{K}{2}, \tag{8.10b}$$

and the solution is given as

$$\begin{aligned} u(x,t) =& \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \\ &+ \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(\xi)d\xi - \int_{x_0}^{x-ct} g(\xi)d\xi\right] \\ =& \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi \end{aligned} \tag{8.11}$$

This solution is called the well-known *d'Alembert solution* of the Cauchy problem for the one-dimensional wave equation.

## Example 27

*Find the solution of the IVP*

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{8.12a}$$

$$u(x, 0) = \sin\ x, \quad x \in \mathbb{R}, \tag{8.12b}$$

$$u_t(x, 0) = \cos\ x, \quad x \in \mathbb{R}. \tag{8.12c}$$

# D'Alembert's Solution for the Homogeneous Wave Equation

It follows from the d'Alembert solution that, if an initial displacement or an initial velocity is located in a small neighborhood of some point $(x_0, t_0)$, it can influence only the area $t > t_0$ bounded by two characteristics $x - ct =$constant and $x + ct =$ constant with slope $\pm(1/c)$ passing through the point $(x_0, t_0)$, as shown in Figure 1. This means that the initial displacement propagates with the speed $\frac{dx}{dt} = c$, whereas the effect of the initial velocity propagates at all speeds up to $c$. This infinite sector $R$ in this figure is called *the range of influence* of the point $(x_0, t_0)$.

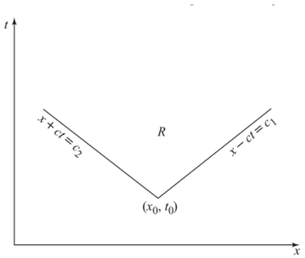


Figure: The Range of Influence of the Point $(x_0, t_0)$.

# D'Alembert's Solution for the Homogeneous Wave Equation

According to (8.11), the value of $u(x_0, t_0)$ depends on the initial data $f$ and $g$ in the interval $[x_0 - ct_0, x_0 + ct_0]$ which is cut out of the initial line by the two characteristics $x - ct$ =constant and $x + ct$ =constant with slope $\pm(1/c)$ passing through the point $(x_0, t_0)$. The interval $[x_0 - ct_0, x_0 + ct_0]$ on the line $t = 0$ is called *the domain of dependence* of the solution at the point $(x_0, t_0)$, as shown in Figure 3.

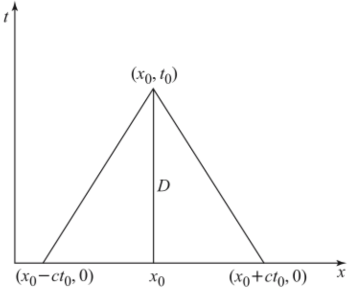


Figure: The Domain of Dependence of the Solution at the Point $(x_0, t_0)$.

## D'Alembert's Solution for the Homogeneous Wave Equation

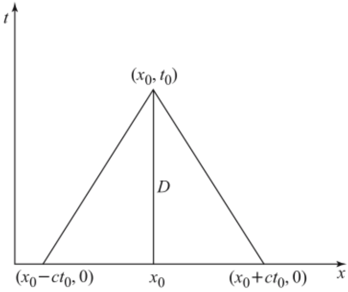


Figure: The Domain of Dependence of the Solution at the Point $(x_0, t_0)$.

Since the solution $u(x,t)$ at every point $(x,t)$ inside the triangular region $D$ in this figure is completely determined by the Cauchy data on the interval $[x_0 - ct_0, x_0 + ct_0]$, the region $D$ is called *the region of determinancy* of the solution.

Now, we consider the Cauchy problem for the nonhomogeneous wave equation

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= h^*(x,t), \quad x \in \mathbb{R}, \quad t > 0, && \text{(8.14a)}\\
u(x,0) &= f(x), \quad x \in \mathbb{R}, && \text{(8.14b)}\\
u_t(x,0) &= g^*(x), \quad x \in \mathbb{R}. && \text{(8.14c)}
\end{aligned}
$$

By coordinate transformation $y = ct$, the problem is reduced to

$$
\begin{aligned}
u_{xx} - u_{yy} &= h(x,y), \quad x \in \mathbb{R}, \quad t > 0, && \text{(8.15a)}\\
u(x,0) &= f(x), \quad x \in \mathbb{R}, && \text{(8.15b)}\\
u_y(x,0) &= g(x), \quad x \in \mathbb{R}. && \text{(8.15c)}
\end{aligned}
$$

where $h \equiv -\frac{h^*}{c^2}$ and $g \equiv \frac{g^*}{c}$.

Let $P_0(x_0, y_0)$ be a point of the plane and let $Q(x_0, 0)$ be the point on the initial line $y = 0$. Then the characteristics, $x \pm y =$constant of (8.15a) are two straight lines drawn through the point $P_0$ with slopes $\pm 1$. Obviously, they intersect the $x$-axis at the points $P_1(x_0 - y_0, 0)$ and $P_2(x_0 + y_0, 0)$, as shown in Figure 4.

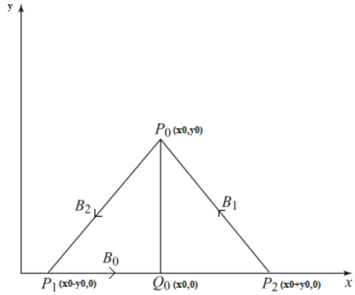


Figure: The Triangular Region $R$.

## D'Alembert's Solution for the Nonhomogeneous Wave Equation

Let the sides of the triangle $P_0P_1P_2$ be designated by $B_0$, $B_1$ and $B_2$, and let $R$ be the region representing the interior of the triangle and its boundaries $B$. Integrating both sides of equation (8.15a), we obtain

$$\int\int_R (u_{xx} - u_{yy})dxdy = \int\int_R h(x,y)dxdy. \qquad (8.16)$$

Now, by taking $M := u_x$ and $N := u_y$ (and, therefore, $u_{xx} = \frac{\partial N}{\partial x}$ and $u_{yy} = \frac{\partial M}{\partial y}$), we apply Green's theorem to obtain

$$\begin{aligned}\int\int_R (u_{xx} - u_{yy})dxdy &= \int\int_R \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) dxdy \\ &= \oint_B (Ndx + Mdy) \qquad (8.17) \\ &= \oint_B (u_y dx + u_x dy).\end{aligned}$$

Since $B = B_0 \cup B_1 \cup B_2$, we have three cases to consider:

- **On $B_0$:** Note that, since $x = x$ where $x$ varies from $x = x_0 - y_0$ to $x = x_0 + y_0$ and $y = 0$ (and, hence, $dx = dx$ and $dy = 0$), we have

$$\begin{aligned} \int_{B_0} (u_x dy + u_y dx) &= \int_{B_0} u_y dx \\ &= \int_{x_0 - y_0}^{x_0 + y_0} u_y(x, 0) dx. \end{aligned} \tag{8.18}$$

Since $B = B_0 \cup B_1 \cup B_2$, we have three cases to consider:

- **On $B_1$:** Note that, since $y = -x + x_0 + y_0$ and $x$ varies from $x = x_0 + y_0$ to $x = x_0$ (and, hence, $dy = -dx$), we have

$$\begin{aligned} \int_{B_1} (u_y dx + u_x dy) &= \int_{B_1} (u_x \cdot (-dx) + u_y \cdot (-dy)) \\ &= \int_{(x,y)=(x_0+y_0,0)}^{(x,y)=(x_0,y_0)} (-du) \\ &= u(x_0 + y_0, 0) - u(x_0, y_0). \end{aligned} \tag{8.19}$$

Since $B = B_0 \cup B_1 \cup B_2$, we have three cases to consider:

- **On $B_2$:** Note that, since $y = x - x_0 + y_0$ and $x$ varies from $x = x_0$ to $x = x_0 - y_0$ (and, hence, $dy = dx$), we have

$$\begin{aligned}\int_{B_2} (u_y dx + u_x dy) &= \int_{B_2} (u_x dx + u_y dy) \\ &= \int_{(x,y)=(x_0,y_0)}^{(x,y)=(x_0-y_0,0)} du \\ &= u(x_0 - y_0, 0) - u(x_0, y_0).\end{aligned} \tag{8.20}$$

# D'Alembert's Solution for the Nonhomogeneous Wave Equation

By (8.17), (8.18), (8.19) and (8.20), we have

$$\begin{aligned}\oint_B (u_y dx + u_x dy) =& -2u(x_0, y_0) + u(x_0 - y_0, 0) \\ &+ u(x_0 + y_0, 0) + \int_{x_0 - y_0}^{x_0 + y_0} u_y(x, 0) dx.\end{aligned} \tag{8.21}$$

Combining (8.16), (8.17) and (8.21), we obtain

$$\begin{aligned}u(x_0, y_0) =& \frac{1}{2}[u(x_0 - y_0, 0) + u(x_0 + y_0, 0)] \\ &+ \frac{1}{2}\int_{x_0 - y_0}^{x_0 + y_0} u_y(x, 0) dx - \frac{1}{2}\int\int_R h(x, y) dx dy.\end{aligned} \tag{8.22}$$

## D'Alembert's Solution for the Nonhomogeneous Wave Equation

Since $x_0$ and $y_0$ are chosen arbitrarily, as a consequence, we replace $x_0$ by $x$ and $y_0$ by $y$ and (8.22) becomes

$$\begin{aligned} u(x,y) =& \frac{1}{2}[f(x-y)+f(x+y)] + \frac{1}{2}\int_{x-y}^{x+y} g(\xi)d\xi \\ & -\frac{1}{2}\int\int_R h(\xi,\eta)d\xi d\eta \end{aligned} \tag{8.23}$$

and replacing $y = ct$, we have

$$\begin{aligned} u(x,t) =& \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(\xi)d\xi \\ & +\frac{1}{2c}\int\int_R h^*(\xi,\tau)d\xi d\tau. \end{aligned} \tag{8.24}$$

The integration region $R$ is a triangular region enclosed by $t = 0$ and characteristics so that it is named as *characteristic triangle*.

$$R = \{(\xi, \tau) \mid 0 \leq \tau \leq t, x - c(t - \tau) < \xi < x + c(t - \tau)\} \quad (8.25)$$

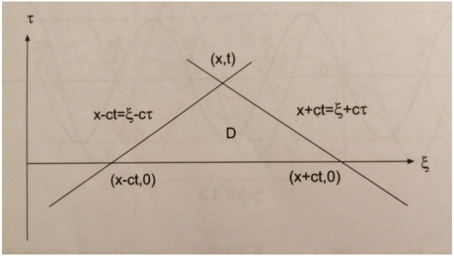


Figure: The Triangular Region $R$.

Therefore, (8.24) yields to

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(\xi)d\xi + \frac{1}{2c}\int_{\tau=0}^{\tau=t}\int_{\xi=x-ct+c\tau}^{\xi=x+ct-c\tau} h^*(\xi,\tau)d\xi d\tau. \tag{8.25}$$

## Example 28

*Determine the solution of*

$$u_{tt} - c^2 u_{xx} = x, \tag{8.26a}$$

$$u(x,0) = \sin\ x, \tag{8.26b}$$

$$u_t(x,0) = x. \tag{8.26c}$$

## Example 29

*Find the solution of the IVP*

$$u_{tt} - c^2 u_{xx} = 0, \quad t > 0, \tag{8.28a}$$

$$u(x,0) = x^3, \tag{8.28b}$$

$$u_t(x,0) = x,. \tag{8.28c}$$

## Example 30

*Find the solution of the IVP*

$$u_{xx} - u_{tt} = 1, \tag{8.30a}$$

$$u(x,0) = \sin\ x, \tag{8.30b}$$

$$u_t(x,0) = x. \tag{8.30c}$$