MTM3502-Partial Differential Equations

Gökhan Göksu, PhD

Week 15

Wave equations can also appear as solutions to physical phenomena that are restricted to bounded regions. Initial data is defined on the finite subinterval $a \leq x \leq b$ of the line $t = 0$ and the values of u and u_x are defined at the endpoints $x = a$ and $x = b$.

Now, consider the following IVP-BVP:

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} =& 0, \quad (0 \leq x \leq \ell,\ t \geq 0) && (12.1a)\\
u(x,0) =& f(x), \quad (0 \leq x \leq \ell) && (12.1b)\\
u_t(x,0) =& g(x), \quad (0 \leq x \leq \ell) && (12.1c)\\
u(0,t) =& 0, \quad (t \geq 0) && (12.1d)\\
u(\ell,t) =& 0. \quad (t \geq 0) && (12.1e)
\end{aligned}
$$

This problem physically corresponds to the vibrating string problem with to fixed endpoints. Since the system is homogeneous, we may choose the solution of the form $u(x,t) = X(x)T(t)$ where X and T are the functions of x and t, respectively. Then, we have

$$XT'' - c^2X''T = 0 \quad \Longrightarrow \quad \frac{T''}{c^2T} = \frac{X''}{X} = \lambda. \tag{12.2}$$

Now, we have three cases to consider.

Cauchy Problem for the Wave Equation

Now, we have three cases to consider:

- **For $\lambda > 0$:** In this case, we have two corresponding ODEs:

$$X'' - \lambda X = 0, \tag{12.3a}$$

$$T'' - \lambda c^2 T = 0. \tag{12.3b}$$

The characteristic equations and their roots for (12.3a) and (12.3b), respectively, yields to the following

$$\begin{aligned} C(r) = r^2 - \lambda = 0 \ &\Rightarrow r_{1,2} = \pm\sqrt{\lambda} \\ &\Rightarrow X(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x} \qquad (12.4a) \\ C(r) = r^2 - \lambda c^2 = 0 \ &\Rightarrow r_{1,2} = \pm c\sqrt{\lambda} \\ &\Rightarrow T(t) = c_3 e^{-c\sqrt{\lambda}t} + c_4 e^{c\sqrt{\lambda}t} \qquad (12.4b) \end{aligned}$$

which, in turn, yields to the solution

$$u(x,t) = \left(c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x}\right)\left(c_3 e^{-c\sqrt{\lambda}t} + c_4 e^{c\sqrt{\lambda}t}\right) \tag{12.5}$$

Cauchy Problem for the Wave Equation

Now, we have three cases to consider:

- **For $\lambda > 0$:** Using (12.1d) to (12.5), we have

$$\begin{aligned} u(0,t) &= (c_1 + c_2)\left(c_3 e^{-c\sqrt{\lambda}t} + c_4 e^{c\sqrt{\lambda}t}\right) = 0 \\ &\Rightarrow c_1 + c_2 = 0, \end{aligned} \tag{12.6}$$

and using (12.1e) to (12.5), we have

$$\begin{aligned} u(\ell,t) &= \left(c_1 e^{-\sqrt{\lambda}\ell} + c_2 e^{\sqrt{\lambda}\ell}\right)\left(c_3 e^{-c\sqrt{\lambda}t} + c_4 e^{c\sqrt{\lambda}t}\right) = 0 \\ &\Rightarrow \left(c_1 e^{-\sqrt{\lambda}\ell} + c_2 e^{\sqrt{\lambda}\ell}\right) = 0. \end{aligned} \tag{12.7}$$

Since

$$\begin{vmatrix} 1 & 1 \\ e^{-\sqrt{\lambda}\ell} & e^{\sqrt{\lambda}\ell} \end{vmatrix} \neq 0 \tag{12.8}$$

the homogeneous system

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{-\sqrt{\lambda}\ell} + c_2 e^{\sqrt{\lambda}\ell} = 0 \tag{12.9}$$

has no solution.

Cauchy Problem for the Wave Equation

Now, we have three cases to consider:

- **For $\lambda = 0$:** In this case, we have two corresponding ODEs:

$$X'' = 0, \tag{12.10a}$$

$$T'' = 0. \tag{12.10b}$$

The general solutions of (12.10a) and (12.10b) yields to

$$u(x,t) = (c_1 x + c_2)(c_3 t + c_4) \tag{12.11}$$

Using (12.1d) to (12.11), we have

$$u(0,t) = c_2 (c_3 t + c_4) = 0 \quad \Longrightarrow \quad c_2 = 0, \tag{12.12}$$

and using (12.1e) to (12.11), we have

$$u(\ell,t) = (c_1 \ell + c_2)(c_3 \ell + c_4) = 0 \quad \Longrightarrow \quad c_1 \ell + c_2 = 0. \tag{12.13}$$

The system

$$c_2 = 0 \quad \text{and} \quad c_1 \ell + c_2 = 0 \tag{12.14}$$

has no nontrivial solution for this case as well.

Now, we have three cases to consider:

- **For $\lambda < 0$:** In this case, by taking $\lambda = -\mu^2$,we have two corresponding ODEs:

$$X'' + \mu^2 X = 0, \tag{12.15a}$$

$$T'' + \lambda c^2 \mu^2 = 0. \tag{12.15b}$$

The characteristic equations and their roots for (12.15a) and (12.15b), respectively, yields to the following

$$\begin{aligned} C(r) = r^2 + \mu^2 = 0 \;&\Rightarrow\; r_{1,2} = \pm i\mu \\ &\Rightarrow\; X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x) \end{aligned} \tag{12.16a}$$

$$\begin{aligned} C(r) = r^2 + c^2\mu^2 = 0 \;&\Rightarrow\; r_{1,2} = \pm ic\mu \\ &\Rightarrow\; T(t) = c_3 \cos(c\mu t) + c_4 \sin(c\mu t) \end{aligned} \tag{12.16b}$$

which, in turn, yields to the solution

$$u(x,t) = (c_1 \cos(\mu x) + c_2 \sin(\mu x))\,(c_3 \cos(c\mu t) + c_4 \sin(c\mu t)) \tag{12.17}$$

Now, we have three cases to consider:

- **For $\lambda < 0$:** Using (12.1d) to (12.17), we have

$$u(0,t) = c_1\left(c_3\cos(c\mu t) + c_4\sin(c\mu t)\right) = 0 \;\Rightarrow\; c_1 = 0, \tag{12.18}$$

and using (12.1e) to (12.18), we have

$$\begin{aligned} u(\ell,t) &= c_2\sin(\mu\ell)\left(c_3\cos(c\mu t) + c_4\sin(c\mu t)\right) = 0 \\ \Rightarrow\; & c_2\sin(\mu\ell) = 0 \\ \Rightarrow\; & \mu\ell = n\pi \quad\Longrightarrow\quad \mu = \frac{n\pi}{\ell}, \text{ for } n \in \mathbb{N}. \end{aligned} \tag{12.19}$$

Replacing $\mu = \frac{n\pi}{\ell}$ into the general solution, we obtain

$$u(x,t) = \sum_{n=1}^{\infty}\left(c_{3n}\cos\left(c\frac{n\pi}{\ell}t\right) + c_{4n}\sin\left(c\frac{n\pi}{\ell}t\right)\right)\sin\left(\frac{n\pi}{\ell}x\right) \tag{12.20}$$

From (12.1b), we have

$$u(x,0) = \sum_{n=1}^{\infty} c_{3n}\sin\left(\frac{n\pi}{\ell}x\right) = f(x). \tag{12.21}$$

Now, we have three cases to consider:

- **For $\lambda < 0$:** Multiplying both sides of (12.21) with $\sin\left(\frac{m\pi x}{\ell}\right)$ and integrating on $[0,\ell]$ yields to

$$
\begin{aligned}
&\int_0^\ell f(x)\sin\left(\frac{m\pi x}{\ell}\right)dx \\
&=\sum_{n=1}^\infty c_{3n}\int_0^\ell \sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx \\
&=-\frac{1}{2}\sum_{n=1}^\infty c_{3n}\int_0^\ell\left(\cos\left(\frac{(n+m)\pi x}{\ell}\right)-\cos\left(\frac{(n-m)\pi x}{\ell}\right)\right)dx \\
&=-\frac{1}{2}\sum_{n=1}^\infty c_{3n}\left(\frac{\sin\left(\frac{(n+m)\pi x}{\ell}\right)}{\frac{(n+m)\pi}{\ell}}-\frac{\sin\left(\frac{(n-m)\pi x}{\ell}\right)}{\frac{(n-m)\pi}{\ell}}\right)\Bigg|_0^\ell \\
&=-\frac{1}{2}\sum_{n=1}^\infty c_{3n}\left(\frac{\sin\left((n+m)\pi\right)}{\frac{(n+m)\pi}{\ell}}-\frac{\sin\left((n-m)\pi\right)}{\frac{(n-m)\pi}{\ell}}\right).
\end{aligned}
\tag{12.22}
$$

Cauchy Problem for the Wave Equation

Now, we have three cases to consider:

- **For $\lambda < 0$:** Considering this result, we have two cases to consider:
  - **For $m \neq n$:** We have

$$-\frac{1}{2}\sum_{n=1}^{\infty} c_{3n}\left(\frac{\sin\left((n+m)\pi\right)}{\frac{(n+m)\pi}{\ell}} - \frac{\sin\left((n-m)\pi\right)}{\frac{(n-m)\pi}{\ell}}\right) = 0 \quad (12.23)$$

  - **For $m = n$:** We have

$$\begin{aligned}
&-\frac{1}{2}\sum_{n=1}^{\infty} c_{3n}\left(\frac{\sin\left((n+m)\pi\right)}{\frac{(n+m)\pi}{\ell}} - \frac{\sin\left((n-m)\pi\right)}{\frac{(n-m)\pi}{\ell}}\right) \\
&= -\frac{1}{2}\sum_{n=1}^{\infty} c_{3n}\left(-\frac{\sin\left((n-m)\pi\right)}{\frac{(n-m)\pi}{\ell}}\right) \qquad (12.24) \\
&= -\frac{1}{2}c_{3m}\left(-\ell\right) \quad (\text{as } (n-m)\pi \to 0)
\end{aligned}$$

Now, we have three cases to consider:

- **For $\lambda < 0$:** Considering these two cases, we get

$$\int_0^\ell f(x)\sin\left(\frac{m\pi x}{\ell}\right)dx = -\frac{1}{2}c_{3m}(-\ell)$$
$$\Rightarrow \quad c_{3m} = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{m\pi x}{\ell}\right)dx \tag{12.25}$$

Deriving (12.20) with respect to t, we have

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{c\pi n}{\ell}\left(-c_{3n}\sin\left(c\frac{n\pi}{\ell}t\right) + c_{4n}\cos\left(c\frac{n\pi}{\ell}t\right)\right)\sin\left(\frac{n\pi}{\ell}x\right). \tag{12.26}$$

and from (12.1c), we obtain

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{c\pi n}{\ell}c_{4n}\sin\left(\frac{n\pi}{\ell}x\right) = g(x). \tag{12.27}$$

Now, we have three cases to consider:

- **For $\lambda < 0$:** Similarly, multiplying both sides of (12.27) with $\sin\left(\frac{m\pi x}{\ell}\right)$ and integrating on $[0, \ell]$ yields to

$$\begin{aligned}
&\int_0^\ell g(x)\sin\left(\frac{m\pi x}{\ell}\right)dx \\
&=\sum_{n=1}^{\infty}\frac{c\pi n}{\ell}c_{4n}\left(-\frac{1}{2}\right)\int_0^\ell -2\sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx \\
&=\sum_{n=1}^{\infty}-\frac{c\pi n}{2\ell}c_{4n}\int_0^\ell\left(\cos\left(\frac{(n+m)\pi x}{\ell}\right)-\cos\left(\frac{(n-m)\pi x}{\ell}\right)\right)dx \\
&=\sum_{n=1}^{\infty}-\frac{c\pi n}{2\ell}c_{4n}\left(\frac{\sin\left(\frac{(n+m)\pi x}{\ell}\right)}{\frac{(n+m)\pi}{\ell}}-\frac{\sin\left(\frac{(n-m)\pi x}{\ell}\right)}{\frac{(n-m)\pi}{\ell}}\right)\Bigg|_0^\ell \\
&=\sum_{n=1}^{\infty}-\frac{c\pi n}{2\ell}c_{4n}\left(\frac{\sin((n+m)\pi)}{\frac{(n+m)\pi}{\ell}}-\frac{\sin((n-m)\pi)}{\frac{(n-m)\pi}{\ell}}\right).
\end{aligned} \tag{12.28}$$

Now, we have three cases to consider:

- **For $\lambda < 0$:** Again, we have two cases to consider
  - **For $m \neq n$:** We have

$$\sum_{n=1}^{\infty} -\frac{c\pi n}{2\ell} c_{4n} \left(\frac{\sin\left((n+m)\pi\right)}{\frac{(n+m)\pi}{\ell}} - \frac{\sin\left((n-m)\pi\right)}{\frac{(n-m)\pi}{\ell}} \right) = 0 \quad (12.29)$$

  - **For $m = n$:** We have

$$\begin{aligned} &\sum_{n=1}^{\infty} -\frac{c\pi n}{2\ell} c_{4n} \left(\frac{\sin\left((n+m)\pi\right)}{\frac{(n+m)\pi}{\ell}} - \frac{\sin\left((n-m)\pi\right)}{\frac{(n-m)\pi}{\ell}} \right) \\ &= \sum_{n=1}^{\infty} -\frac{c\pi n}{2\ell} c_{4n} \left(-\frac{\sin\left((n-m)\pi\right)}{\frac{(n-m)\pi}{\ell}} \right) \qquad (12.30) \\ &= -\frac{c\pi m}{2\ell} c_{4m} (-\ell) \quad (\text{as } (n-m)\pi \to 0) \end{aligned}$$

Cauchy Problem for the Wave Equation

Now, we have three cases to consider:

- **For $\lambda < 0$:** Again from these two cases, we get

$$\int_0^\ell g(x)\sin\left(\frac{m\pi x}{\ell}\right)dx = \frac{c\pi m}{2}c_{4m}$$
$$\Rightarrow \quad c_{4m} = \frac{2}{c\pi m}\int_0^\ell g(x)\sin\left(\frac{m\pi x}{\ell}\right)dx \tag{12.31}$$

Combining (12.20), (12.25) and (12.31), we have the particular solution to the IVP-BVP (12.1a)-(12.1e):

$$u(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{m\pi x}{\ell}\right)dx\right)\cos\left(c\frac{n\pi}{\ell}t\right) + \left(\frac{2}{c\pi n}\int_0^\ell g(x)\sin\left(\frac{n\pi x}{\ell}\right)dx\right)\sin\left(c\frac{n\pi}{\ell}t\right)\right] \cdot \sin\left(\frac{n\pi}{\ell}x\right). \tag{12.32}$$

Suppose that, we have a rod with a length ℓ and the heat is assumed to be distributed equally over the cross section at time t. The surface of the rod is insulated, and therefore, there is no heat loss through the boundary. The temperature distribution of the rod is given by the solution of the IVP-BVP:

$$u_t - \kappa u_{xx} = 0, \quad (0 \leq x \leq \ell,\ t \geq 0) \tag{12.33a}$$

$$u(0,t) = 0, \quad (t \geq 0) \tag{12.33b}$$

$$u(\ell,t) = 0, \quad (t \geq 0) \tag{12.33c}$$

$$u(x,0) = f(x), \quad (0 \leq x \leq \ell) \tag{12.33d}$$

Cauchy Problem for the Heat Conduction Problem

Here, $u(x,t)$ represents the heat at the position x and the time t and κ is a thermal conductivity constant, which is determined by experiments and depends on the material. Since (12.33a) is homogeneous, we may seek for a solution of type $u(x,t) = X(x)T(t)$ with functions X and T which yields to

$$\frac{X''}{X} = \frac{T'}{\kappa T} = -\mu^2. \tag{12.34}$$

and this gives us two ODEs:

$$X'' + \mu^2 X = 0, \tag{12.35a}$$

$$T' + \mu^2 \kappa T = 0. \tag{12.35b}$$

The BVs associated to (12.33b) and (12.33c) can be obtained

$$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0, \tag{12.36a}$$

$$u(\ell,t) = X(\ell)T(t) = 0 \implies X(\ell) = 0. \tag{12.36b}$$

From these BVs, the nontrivial solution for X can be obtained as

$$X_n(x) = c_{1n} \sin\left(\frac{n\pi x}{\ell}\right), \quad n \in \mathbb{N}. \tag{12.37}$$

On the other hand, the general solution of (12.35b) will be

$$T_n(t) = c_{2n} e^{-\left(\frac{n\pi}{\ell}\right)^2 \kappa t}, \quad n \in \mathbb{N}. \tag{12.38}$$

The general solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{\ell}\right)^2 \kappa t} \sin\left(\frac{n\pi x}{\ell}\right), \tag{12.39}$$

where $c_n = c_{1n} c_{2n}$.

The latter BV (12.33d) enables us to calculate the Fourier coefficients (c_n's). To that aim, we first evaluate the following

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x), \tag{12.40}$$

and, then, multiply both sides of (12.40) with $\sin\left(\frac{m\pi x}{\ell}\right)$ and integrating on $[0,\ell]$ yields to

$$\begin{aligned} \sum_{n=1}^{\infty} c_n \int_0^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) dx \\ = \int_0^{\ell} f(x) \sin\left(\frac{m\pi x}{\ell}\right) dx. \end{aligned} \tag{12.41}$$

Observing the identity[1]

$$\int_0^\ell \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\ell}{2}, & \text{if } m = n \end{cases} \tag{12.42}$$

and using it in (12.41), we have

$$\begin{aligned} c_m\frac{\ell}{2} &= \int_0^\ell f(x)\sin\left(\frac{m\pi x}{\ell}\right)dx \\ \Rightarrow\ c_m &= \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{m\pi x}{\ell}\right)dx \end{aligned} \tag{12.43}$$

[1] Recall the “orthogonality of basis functions" discussion of last week. The same implication can be revised by a simple variable change $x' = \left(\frac{\ell}{2\pi}\right)x$.

In conclusion, the solution for the IVP-BVP (12.33a)-(12.33d) will be

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{2}{\ell}\int_0^{\ell}f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx\right)e^{-\left(\frac{n\pi}{\ell}\right)^2\kappa t}\sin\left(\frac{n\pi x}{\ell}\right), \tag{12.44}$$

which is the required solution.

Exercise 32 (Cauchy Problem for Wave Equation)

Solve the following Cauchy problem:

$$
\begin{aligned}
u_{tt} - 4u_{xx} &= 0, \quad (0 \le x \le 1,\ t \ge 0) && (13.1a)\\
u(x,0) &= \sin(2\pi x), \quad (0 \le x \le 1) && (13.1b)\\
u_t(x,0) &= 1, \quad (0 \le x \le 1) && (13.1c)\\
u(0,t) &= 0, \quad (t \ge 0) && (13.1d)\\
u(1,t) &= 0. \quad (t \ge 0) && (13.1e)
\end{aligned}
$$

Solution:

$$
\begin{aligned}
u(x,t) = {} & \cos(4\pi t)\sin(2\pi x)\\
& + \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2\pi^2} \sin((4n+2)\pi t)\sin((2n+1)\pi x) \qquad (13.4)
\end{aligned}
$$

Exercise 33 (Cauchy Problem for Heat Conduction Problem)

Solve the following Cauchy problem:

$$
\begin{aligned}
u_t - 2u_{xx} &= 0, \quad (0 \le x \le 3,\ t \ge 0) && (13.5a)\\
u(0,t) &= 0, \quad (t \ge 0) && (13.5b)\\
u(3,t) &= 0, \quad (t \ge 0) && (13.5c)\\
u(x,0) &= x(3-x), \quad (0 \le x \le 3) && (13.5d)
\end{aligned}
$$

Solution:

$$u(x,t) = \frac{72}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} e^{-\left(\frac{(2n+1)\pi}{3}\right)^2 t} \sin\left(\frac{(2n+1)\pi x}{3}\right) \quad (13.7)$$