MTM3502-Partial Differential Equations

Gökhan Göksu, PhD

Week 2

Recall the first order linear PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \tag{1.8}$$

Let us define the differential operator

$$L = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial x} + c(x,y) \tag{2.1}$$

which enables us to rewrite (1.8), in short, as

$$Lu = d(x,y). \tag{2.2}$$

Any operator is said to be linear if it satisfies the following, for all $c_1, c_2 \in \mathbb{R}$ and all partially differentiable functions f_1 and f_2,

$$L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2). \tag{2.3}$$

Definition

If $d(x, y) \equiv 0$ in (1.8), then the corresponding equation

$$Lu = 0 \tag{2.4}$$

is called the *homogeneous equation* corresponding to (2.3). Moreover, if $u = \phi(x, y)$ is a *particular solution* of (1.8), then it satisfies the following identity:

$$a(x, y)\phi_x + b(x, y)\phi_y + c(x, y)\phi \equiv d(x, y) \tag{2.5}$$

Definition

The partially differentiable function $u = \phi(x, y)$ is said to be the *integral surface* of (1.8), if it satisfies (1.8).

Definition

Any family of surfaces is called the *general solution of the homogeneous equation* if it contains an arbitrary function which satisfies (2.4).

Definition

Let u_h be the general solution of the homogeneous equation (2.4) and u_p be the particular solution of (2.3). Then, $u = u_h + u_p$ is called the *general solution of the nonhomogeneous equation*.

Example

Consider the PDE $u_x + u = x$. The corresponding homogeneous PDE is $u_x + u = 0$. Multiplying both sides with e^x yields to

$$e^x u_x + e^x u = \frac{\partial}{\partial x}(e^x u) = 0.$$

Integrating both sides, we have

$$e^x u = \phi(y) \implies u_h = e^{-x}\phi(y),$$

where ϕ is any arbitrary differentiable function. This is also called an integral surface for the homogeneous equation.

Example

Consider the PDE

$$u_x + u = x$$

Searching a particular solution for the nonhomogeneous equation of the form $u_p = ax + b$ will give us $a = 1$ and $b = -1$. This tells us that the general solution of the nonhomogeneous equation can be obtained as

$$u = e^{-x}\phi(y) + x - 1,$$

where ϕ is any arbitrary differentiable function.

Now, let us see a systematic way to find the first order linear PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \tag{1.8}$$

Let $D \subset \mathbb{R}^2$ be a simply connected region of $\mathbb{R}^2$ and suppose that $a(x,y)^2 + b(x,y)^2 \neq 0$ with $a(x,y) = 0$ or $b(x,y) = 0$ on D.

In this case, the solution can be obtained using the same solution methodology as the first order linear nonhomogeneous ordinary differential equations (ODEs). Naturally, if there is no partial derivative with respect to any variable in the equation, that variable is evaluated as a constant in the integration and the function of this variable is replaced by the constant of integration in ODEs.

Without loss of generality, let us take $b(x,y) \equiv 0$ and $a(x,y) \neq 0$. Then, (1.8) have the form

$$u_x + \frac{c(x,y)}{a(x,y)} u = \frac{d(x,y)}{a(x,y)} \tag{2.6}$$

Then, the integrating factor will be

$$\mu(x,y) = e^{\int \frac{c(x,y)}{a(x,y)} dx}. \tag{2.7}$$

Multiplying both sides of (2.6) with (2.7), we obtain

$$e^{\int \frac{c(x,y)}{a(x,y)} dx} u_x + \frac{c(x,y)}{a(x,y)} u e^{\int \frac{c(x,y)}{a(x,y)} dx} = \frac{d(x,y)}{a(x,y)} e^{\int \frac{c(x,y)}{a(x,y)} dx}. \tag{2.8}$$

Note that, we obtained an exact differential in the left hand side so we can, thus, rewrite (2.8) as

$$\frac{\partial}{\partial x}\left[u e^{\int \frac{c(x,y)}{a(x,y)} dx}\right] = \frac{d(x,y)}{a(x,y)} e^{\int \frac{c(x,y)}{a(x,y)} dx}. \tag{2.9}$$

Integrating (2.8), we have

$$ue^{\int \frac{c(x,y)}{a(x,y)}dx} = \int \frac{d(x,y)}{a(x,y)} \left[e^{\int \frac{c(x,y)}{a(x,y)}dx} \right] dx + \phi(y), \tag{2.10}$$

which, in turn, yields to the solution

$$u(x,y) = e^{-\int \frac{c(x,y)}{a(x,y)}dx} \left\{ \int \frac{d(x,y)}{a(x,y)} \left[e^{\int \frac{c(x,y)}{a(x,y)}dx} \right] dx + \phi(y) \right\}, \tag{2.11}$$

where ϕ is any differentiable arbitrary function.

In the case that $a(x,y) \equiv 0$ and $b(x,y) \neq 0$, then the equation will be

$$u_y + \frac{c(x,y)}{b(x,y)} u = \frac{d(x,y)}{b(x,y)} \tag{2.12}$$

and its solution can be obtained similarly as

$$u(x,y) = e^{-\int \frac{c(x,y)}{b(x,y)}dy} \left\{ \int \frac{d(x,y)}{b(x,y)} \left[e^{\int \frac{c(x,y)}{b(x,y)}dy} \right] dy + \varphi(x) \right\}, \tag{2.13}$$

where φ is any differentiable arbitrary function.

WARNING: Do not try to memorize the solution form, but try to obtain it using the proposed method!!!

Example

Find the general solution of the following PDE

$$xy^3u_x + x^2y^3u = e^{-\frac{x^2}{2}} \cos 2y.$$

Example

Find the general solution of the following PDE

$$\sqrt{3x^2 + y^2}u_y + (3y\cos 2x)u = 2x^2 y\cot 2x.$$

The Method of Integrating Factors for FO Linear PDEs

Now, let's examine the presence of both derivatives in the first order linear PDEs:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \tag{1.8}$$

Define, first, the following transformation

$$\xi = \xi(x,y), \ \eta = \eta(x,y). \tag{2.14}$$

Nonsingularity of the Jacobian matrix, i.e.

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0, \tag{2.15}$$

guarantees the existence of an invertible transformation. Calculating the partial derivatives using this transformation yields to

$$\begin{aligned} p =& u_x = u_\xi \xi_x + u_\eta \eta_x, \\ q =& u_y = u_\xi \xi_y + u_\eta \eta_y. \end{aligned} \tag{2.16}$$

Using (2.16) into (1.8), we have

$$(a(x,y)\xi_x + b(x,y)\xi_y)u_\xi + (a(x,y)\eta_x + b(x,y)\eta_y)u_\eta + c(x,y)u = d(x,y). \tag{2.17}$$

Without loss of generality, let us pick η to make

$$a(x,y)\eta_x + b(x,y)\eta_y = 0 \ \Rightarrow\ \frac{\eta_x}{\eta_y} = -\frac{b(x,y)}{a(x,y)},\ (\eta_y \neq 0). \tag{2.18}$$

On the other hand, let us fix the **characteristic coordinate** η as

$$\eta = \eta(x,y) = \text{const} \ \Rightarrow d\eta = \eta_x dx + \eta_y y = 0 \\ \Rightarrow \frac{\eta_x}{\eta_y} = -\frac{dy}{dx}, \tag{2.19}$$

which, in turn, yields to the following ODE

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}. \tag{2.20}$$

Solving the ODE (2.20) enables us to determine $\eta = \eta(x, y)$. Once $\eta = \eta(x, y)$ is determined, $\xi = \xi(x, y)$ can be arbitrarily chosen to be linearly independent. For simplicity, choosing $\xi = \xi(x, y) = x$ may often be sufficient. Naturally, when making this special choice, it should be ensured that it is linearly independent of $\eta = \eta(x, y)$. The transformation

$$\xi = x, \ \eta = \eta(x, y), \ \frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0, \tag{2.21}$$

will transform the PDE (1.8) into

$$A(\eta, \xi)u_\xi + C(\eta, \xi)u = D(\eta, \xi) \tag{2.22}$$

where $A(\eta, \xi)$, $C(\eta, \xi)$ and $D(\eta, \xi)$ are the functions after transformation.

Solving (2.22) will give the solution in terms of η and ξ as

$$u(\eta,\xi) = e^{-\int \frac{C(\eta,\xi)}{A(\eta,\xi)}d\xi}\left\{\int \frac{D(\eta,\xi)}{A(\eta,\xi)}\left[e^{\int \frac{C(\eta,\xi)}{A(\eta,\xi)}d\xi}\right]d\xi + \phi(\eta)\right\}, \tag{2.23}$$

where $\phi(\eta)$ is any differentiable arbitrary function. Once the general solution (2.23) is obtained in terms of η and ξ, the inverse transformation obtained from (2.14) will give the general solution in terms of x and y.

Example

Find the general solution of the following PDE

$$xyp - x^2q + yu = 0,$$

$(p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y})$.

Example

Find the general solution of the following PDE

$$4xp - 8yq + 4u = x\cos x,$$

$(p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y})$.

Example

Find the general solution of the following PDE

$$(2xy^3 - x^4)p + (2y^4 - x^3y)q + 2(2y^3 - x^3)u = x + y^2,$$

$(p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y})$.