MTM3502-Partial Differential Equations

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Week 5

Solution of Nonlinear PDEs of First Order: The Method of Charpit

Consider the first order nonlinear PDE

$$F(x, y, u, p, q) = 0. \tag{4.1}$$

The method of Charpit consists of finding another first order PDE of the form

$$G(x, y, u, p, q) = 0 \tag{4.2}$$

so that

- The equations (4.1) and (4.2) can be solved for p and q and
- $du = pdx + qdy$ is integrable.

From the compatibility of (4.1) and (4.2), we have

$$\frac{\partial(F,G)}{\partial(x,p)} + \frac{\partial(F,G)}{\partial(y,q)} + p\frac{\partial(F,G)}{\partial(u,p)} + q\frac{\partial(F,G)}{\partial(u,q)} \tag{4.3}$$

which, in turn, yields

$$\begin{aligned} F_p\frac{\partial G}{\partial x} + F_q\frac{\partial G}{\partial y} + (pF_p + qf_q)\frac{\partial G}{\partial u} \\ - (F_x + pF_u)\frac{\partial G}{\partial p} - (F_y + qF_u)\frac{\partial G}{\partial q} = 0. \end{aligned} \tag{4.4}$$

Therefore, the corresponding auxiliary equations are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{dp}{-(F_x + pF_u)} = \frac{dq}{-(F_y + qF_u)}. \quad (4.5)$$

If we can find any solution of (4.5) involving p and q, it will serve as a PDE given by (4.2). Solving (4.1) and (4.2) for p and q, we can get the integral of $du = pdx + qdy$ as the required solution.

Example 14

Use the method of Charpit to solve PDE

$$(p^2 + q^2)y = qu. \tag{4.6}$$

Example 15

Use the method of Charpit to solve the PDE

$$u^2 = pqxy. \tag{4.14}$$

In this subsection, we will discuss a geometrical method for solving nonlinear first order PDEs which is known as Cauchy's Method of Characteristics.

Recall that the plane passing through the point $P(x, y, u)$ with its normal parallel to the direction $\vec{n}$ defined by the direction ratios $\left[p_0, q_0, -1\right]$ is uniquely specified by the 5-tuple $(x_0, y_0, u_0, p_0, q_0)$ and the 5-tuple (x, y, u, p, q) is called the plane element of the space. In particular, a plane element whose components satisfy the equation

$$F(x, y, u, p, q) = 0. \tag{1.17}$$

is called an integral element of the equation (1.17). Using (1.17), we can obtain q, by fixing the values of x, y, u and p, as

$$q = G(x, y, u, p). \tag{4.21}$$

By fixing $x = x_0$, $y = y_0$ and $u = u_0$, we obtain the plane element $(x_0, y_0, u_0, p, G(x_0, y_0, u_0, p))$ which depends on p. This element envelopes a cone with the vertex P, named as *elementary cone* (or *Monge's cone*) and can be demonstrated as below.

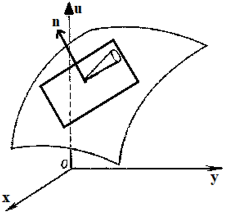

Figure: The Elementary Cone of the PDE (1.17) at P.

In order to solve the PDE (1.17), we find the characteristics equation for which we define by a curve C given by

$$x = x(t),\ y = y(t),\ u = u(t). \tag{4.22}$$

Each point of this curve touches a generator (the edge) of the elementary cone and the strip so formed in this curve is known as a characteristic strip.

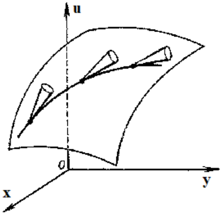

Figure: The Characteristic Strip of the PDE (1.17).

The point $(x+dx, y+dy, u+du)$ lies in the tangent plane to the elementary cone at P, if

$$du = pdx + qdy \tag{4.23}$$

where p and q satisfy the relation (1.17). Differentiating (1.17) and (4.23) with respect to p, we obtain

$$\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial p} = F_p + F_q\frac{\partial q}{\partial p} = 0, \tag{4.24}$$

and

$$dx + \frac{\partial q}{\partial p}dy = 0, \tag{4.25}$$

respectively. Solving (4.24) and (4.25) for dx, dy and du, we obtain

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q}. \tag{4.26}$$

Therefore, along the chracteristic strip, the functions $x'(t)$, $y'(t)$ and $u'(t)$ are proportional to F_p, F_q and $pF_p + qF_q$, respectively. Thus, equating (4.26) with dt, we have

$$x'(t) = F_p, \; y'(t) = F_q, \; u'(t) = pF_p + qF_q. \tag{4.27}$$

We also have $p = p(x, y)$ and $q = q(x, y)$ where x and y are functions of t. Differentiating p with respect to t, we get

$$p'(t) = \frac{\partial p}{\partial x} x'(t) + \frac{\partial p}{\partial y} y'(t) = p_x F_p + p_y F_q. \tag{4.28}$$

Note that $q_x = p_y$, so that we have

$$p'(t) = p_x F_p + q_x F_q. \tag{4.29}$$

Differentiating (1.17) with respect to x, we get

$$\begin{aligned}\frac{dF}{dx} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} \\ &= F_x + pF_u + F_p p_x + F_q q_x \\ &= 0.\end{aligned} \tag{4.30}$$

Solving (4.29) and (4.30) leads to

$$F_x + pF_u + p'(t) = 0 \implies p'(t) = -(F_x + pF_u). \tag{4.31}$$

Similarly, we find that

$$F_y + qF_u + q'(t) = 0 \implies q'(t) = -(F_y + qF_u). \tag{4.32}$$

Finally, we have the following system of ODEs for the determination of the characteristic strip.

$$\begin{aligned} x'(t) =& F_p,\ y'(t) = F_q,\ u'(t) = pF_p + qF_q, \\ p'(t) =& -(F_x + pF_u),\ q'(t) = -(F_y + qF_u). \end{aligned} \tag{4.33}$$

In order to solve the particular solution of the given PDE (1.17), we will now introduce the Cauchy problem. A Cauchy problem asks for the solution of a PDE that satisfies certain conditions that are given on a hypersurface in the domain. A Cauchy problem can be an initial value problem (IVP) and/or a boundary value problem (BVP).

Suppose that, we want to find the particular solution of (1.17), which passes through a curve Γ' (which is considered as the initial values) whose parametric equations are

$$\Gamma' \; : \; x = x_0(s), \; y = y_0(s), \; u = u_0(s) \tag{4.34}$$

which is demonstrated below.

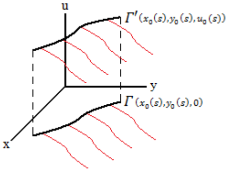

Figure: The Cauchy Problem of the PDE (1.17).

Note that, $u = u_0(s)$ is called as initial condition (IC) and $x = x_0(s)$ and $y = y_0(s)$ are called as boundary conditions (BCs).

The initial values p_0 and q_0 are determined from the relations

$$u_0'(s) = p_0(s)x_0'(s) + q_0(s)y_0'(s), \tag{4.35a}$$
$$F(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s)) = 0. \tag{4.35b}$$

If we substitute the values x_0, y_0, u_0, p_0 and q_0 for $t = 0$ in the solution obtained from the characteristic equations (4.33), we obtain

$$x = x(s,t),\ \ y = y(s,t),\ \ u = u(s,t). \tag{4.36}$$

The elimination of s and t from these equations provides us a relation in terms of x, y and u

$$\phi(x, y, u) = 0 \tag{4.37}$$

which is the required integral surface of the given equation through the given curve Γ'.

Example 16

Find the characteristics of the equation $pq = u$ and determine the integral surface which passes through the parabola $x = 0$, $y^2 = u$.

Example 17

Determine the characteristics of the equation $u = p^2 - q^2$ *and find the integral surface which passes through the parabola* $4u + x^2 = 0$, $y = 0$ ($p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$).