MTM3502-Partial Differential Equations

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Week 9

The general form of a second order linear PDE is given as

$$\begin{aligned} &A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} \\ &+ D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \end{aligned} \tag{5.1}$$

where the coefficients A, B, C, D, E, F and G are the functions of x and y. The classification of a second order linear PDE is suggested by the classification of the quadratic equation of conic sections in analytic geometry. The PDE is said to be *hyperbolic*, *parabolic* or *elliptic* at a point (x_0, y_0) as

$$B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0) \tag{5.2}$$

is positive, zero or negative.

If (5.2) is positive, zero or negative at all points, then the PDE is said to be hyperbolic, parabolic or ellyptic in a domain. In case of two independent variables, it is always possible to reduce the given equation into canonical form in a given domain, which is not possible for several independent variables. Let us consider the following transformation

$$\xi = \xi(x, y), \quad \eta = \eta(x, y) \tag{5.3}$$

with sufficiently smooth functions ξ and η. Note that, if the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \tag{5.4}$$

is nonzero in the region, then the transformation is well-defined and x and y can be determined uniquely from (5.3).

Then, we have

$$
\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x, \quad u_y = u_\xi \xi_y + u_\eta \eta_y, \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}, \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y \\
&\quad + u_\xi \xi_{xy} + u_\eta \eta_{xy}, \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy},
\end{aligned}
\tag{5.5}
$$

Substituting these values in (5.1), we obtain

$$
\begin{aligned}
&A^*(\xi,\eta)u_{\xi\xi} + B^*(\xi,\eta)u_{\xi\eta} + C^*(\xi,\eta)u_{\eta\eta} \\
&+ D^*(\xi,\eta)u_\xi + E^*(\xi,\eta)u_\eta + F^*(\xi,\eta)u = G^*(\xi,\eta)
\end{aligned}
\tag{5.6}
$$

where

$$
\begin{aligned}
A^*(\xi,\eta) =& A(\xi,\eta)\xi_x^2 + B(\xi,\eta)\xi_x\xi_y + C(\xi,\eta)\xi_y^2,\\
B^*(\xi,\eta) =& 2A(\xi,\eta)\xi_x\eta_x + B(\xi,\eta)(\xi_x\eta_y + \xi_y\eta_x)\\
&+ 2C(\xi,\eta)\xi_y\eta_y,\\
C^*(\xi,\eta) =& A(\xi,\eta)\eta_x^2 + B(\xi,\eta)\eta_x\eta_y + C(\xi,\eta)\eta_y^2,\\
D^*(\xi,\eta) =& A(\xi,\eta)\xi_{xx} + B(\xi,\eta)\xi_{xy} + C(\xi,\eta)\xi_{yy}\\
&+ D(\xi,\eta)\xi_x + E(\xi,\eta)\xi_y,\\
E^*(\xi,\eta) =& A(\xi,\eta)\eta_{xx} + B(\xi,\eta)\eta_{xy} + C(\xi,\eta)\eta_{yy}\\
&+ D(\xi,\eta)\eta_x + E(\xi,\eta)\eta_y,\\
F^*(\xi,\eta) =& F(\xi,\eta), \quad G^*(\xi,\eta) = G(\xi,\eta)
\end{aligned}
\tag{5.7}
$$

Note that, (5.6) has the same form as the original (5.1). Therefore, we can say that the nature of the equation remains invariant under such a transformation if the Jacobian does not vanish. This lies on the fact that the *discriminant* does not change under transformation

$$\begin{aligned} B^{*2}(\xi,\eta) &- 4A^{*}(\xi,\eta)C^{*}(\xi,\eta) \\ &= J^2(\xi,\eta)(B^2(\xi,\eta) - 4A(\xi,\eta)C(\xi,\eta)) \text{ or} \\ B^{*2}(x,y) &- 4A^{*}(x,y)C^{*}(x,y) \\ &= J^2(x,y)(B^2(x,y) - 4A(x,y)C(x,y)) \end{aligned} \tag{5.8}$$

Before we go further, we emphasize that we sometimes write the functions by omitting the dependence of (x, y) or (ξ, η) which can be clearly understood from the context. As an example, we may write (5.8), in short as

$$B^{*2} - 4A^*C^* = J^2(B^2 - 4AC). \tag{5.9}$$

The classification of (5.1) depends on the functions A, B and C at a given point (x, y). We, therefore, may rewrite (5.1) as

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y) \tag{5.10}$$

and rewrite (5.6) as

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta). \tag{5.11}$$

In mathematics, a canonical form of a mathematical object is a standard way of presenting that object as a mathematical expression. Often, it is one which provides the simplest representation of an object and which allows it to be identified in a unique way. In most fields, a canonical form specifies a unique representation for every object, without the requirement of uniqueness of the representation.

Considering a second order linear PDE, reducing a PDE into a canonical form means that the second order terms of the PDE is represented in terms of either $u_{\xi\eta}$ or $u_{\xi\xi}$ and $u_{\eta\eta}$ to represent the cases $B^{*2} - 4A^*C^* > 0$, $B^{*2} - 4A^*C^* = 0$ and $B^{*2} - 4A^*C^* < 0$. We, therefore, analyze the canonical forms in three subsections.

Second Order Linear PDEs: Canonical Forms of Hyperbolic PDEs

We, firstly, consider the case that $B^2 - 4AC > 0$. Note that, under coordinate change with a non-vanishing Jacobian, the determinant yields to $B^{*2} - 4A^*C^* > 0$ (See (5.9)). We have two cases to obtain canonical forms of hyperbolic PDEs:

- $A^* = C^* = 0$ and $B^* \neq 0$,
- $C^* = -A^* \neq 0$ and $B^* = 0$.

Considering the case that $A^* = C^* = 0$ and $B^* \neq 0$, we are able to write the PDE in terms of $u_{\xi\eta}$:

$$\begin{aligned} A^* =& A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\ &\implies A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0, \\ C^* =& A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \\ &\implies A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0. \end{aligned} \tag{5.12}$$

Along the curves $\xi =$constant and $\eta =$constant, we have $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$ and $\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$ (note that $d\xi = \xi_x dx + \xi_y dy = 0$ and $d\eta = \eta_x dx + \eta_y dy = 0$). From this fact and the roots of (5.12), we have

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{5.13}$$

which are called as *characteristic equations*. These equations are ODEs for families of curves in the xy-plane along which $\xi =$ constant and $\eta =$constant and the integrals of these equations are called the *characteristic curves*. The solutions, therefore, may be written as

$$\phi_1(x, y) = c_1 \text{ and } \phi_2(x, y) = c_2, \text{ for } c_1, \ c_2 \text{ are constants.} \tag{5.14}$$

Hence the transformations

$$\xi = \phi_1(x, y) \text{ and } \eta = \phi_2(x, y) \tag{5.15}$$

will transform (5.10) into a canonical form.

Since $B^2 - 4AC > 0$, the integration of (5.13) yield to two real and distinct families of characteristics. The equation (5.11) reduces to

$$u_{\xi\eta} = H_1 \tag{5.16}$$

where $H_1 = \frac{H^*}{B^*}$ (note that $B^* \neq 0$). This form is called the *first canonical form of the hyperbolic PDEs*.

Similarly, for $C^* = -A^* \neq 0$ and $B^* = 0$, we are able to write the PDE in terms of $u_{\xi\xi}$ and $u_{\eta\eta}$

$$u_{\xi\xi} - u_{\eta\eta} = H_2 \tag{5.17}$$

which is called the *second canonical form of the hyperbolic PDEs* where $H_2 = \frac{H^*}{A^*}$ (note that $A^* \neq 0$).

Example 18

Find canonical form of the PDE

$$y^2 u_{xx} - x^2 u_{yy} = 0. \tag{5.18}$$