MTM3691-Theory of Linear Programming

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Week 2

Example (The Reddy Mikks Company)

Reddy Mikks produces interior and exterior paints from two raw materials, M_1 and M_2. The basic data for the problem is presented in the table below:

	Raw material usage per ton (tons)		Daily maximum availability (tons)
	Exterior paint	Interior paint	
Raw material, M_1	6	4	24
Raw material, M_2	1	2	6
Profit per ton (1000 TL)	5	4	

Additionally, a market survey has determined that the daily interior paint demand is at most 2 tons. It was also found in the same research that the daily interior paint demand cannot exceed that for daily exterior paint demand by 1 ton. Reddy Mikks wants to determine the optimum production quantities of exterior and interior paints to maximize their daily profit.

The LP model has three fundamental components:

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-
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The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

After defining the decision variables, the objective function is to be defined using these definitions. The goal for interior and exterior painting will be to increase the daily profit from interior and exterior painting (i.e. to maximize profit). If we denote that z represents the daily profit, the goal will be

Next, we construct the constraints that restricts raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix} \text{Usage of a raw material} \\ \text{by both paints} \end{pmatrix} \leq \begin{pmatrix} \text{Maximum raw material} \\ \text{availability} \end{pmatrix}.$$

From the data given in the problem setting, we have the following:

Usage of raw material $M_1 =$

Usage of raw material $M_2 =$

Since the daily availabilities of the raw materials M_1 and M_2 are limited to 24 and 6 tons, respectively, the associated restrictions are given as

Moreover, there are further constraints to be considered.

1) Demand for interior paint is limited to maximum 2 tons per day:

2) The excess of the daily production of interior over exterior paint should not exceed 1 tons per day:

3) The **non-negativity** constraint, which is not directly included in the model and states that variables cannot take negative values, should also be added to the model:

Any values of x_1 and x_2 that satisfy all five constraints above constitute a _____. Otherwise, the solution is _____. For example, the solution $x_1 = 3$ tons per day and $x_2 = 1$ tons per day is _____. On the other hand the solution $x_1 = 4$ and $x_2 = 1$ is _____. The goal of the problem to find _____, or the _____, that maximizes the total profit.

In this example, the objective functions and the constraints are all linear. The linearity assumption is equivalent to the following two properties:

1) **(Proportionality):** This feature states that the effect of each decision variable on both the objective function and all constraints should be directly proportional to the value of the corresponding variable. For example in Reddy Mikks problem, if there is a discount when the sales exceed certain values, the revenues will not be proportional to the increase in the sales amount.

2) **(Additivity):** This feature stipulates that the total contribution of all variables in the constraints and the objective function consists of the sum of the individual contributions of these variables. For example, if there are two *competing products*, this feature will not be realized, as increasing the sales of one of them will adversely affect the sales of the other.

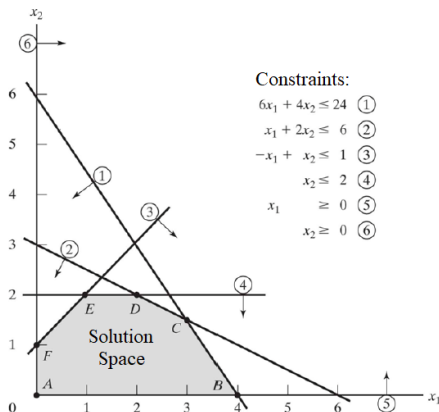The graphical procedure includes two steps:

1) Determination of the feasible solution space,
2) Determining the optimum solution among all the feasible points in the feasible solution space.

The method is explained for cases where the objective function is both maximum and minimum.

Example

*Let us consider the Reddy Mikks problem again.*

- **Step 1 (Determination of the Feasible Solution Space):**

Constraints:

$$6x_1 + 4x_2 \leq 24 \quad \text{①}$$
$$x_1 + 2x_2 \leq 6 \quad \text{②}$$
$$-x_1 + x_2 \leq 1 \quad \text{③}$$
$$x_2 \leq 2 \quad \text{④}$$
$$x_1 \geq 0 \quad \text{⑤}$$
$$x_2 \geq 0 \quad \text{⑥}$$

- **Step 2 (Determination of the Optimum Solution):** The solution space is formed by line segments connecting the vertices *A*, *B*, *C*, *D*, *E* and *F*. Any point on or within the boundaries of *ABCDEF* space is a feasible solution that satisfies all constraints. Since *ABCDEF* feasible space contains an infinite number of suitable points, a systematic method is needed to find the optimum point.

  In order to determine the optimum solution, the direction of increase of the objective function in the form $z = 5x_1 + 4x_2$ (since the goal is to be maximized) must be known. This can be done in practice by arbitrarily increasing the values of z. For example, for the levels $z = 10$, $z = 15$ and $z = 21$, the lines $5x_1 + 4x_2 = 10$, $5x_1 + 4x_2 = 15$ and $5x_1 + 4x_2 = 21$ are the equivalent level lines. The inclusion of this to the solution of the model and the direction of increase of z can be determined as shown in the following figure.

(Max $z = 5x_1 + 4x_2$)

increase in z

$z = 21$

$z = 15$

$z = 10$

$x_1 + 2x_2 \leq 6$
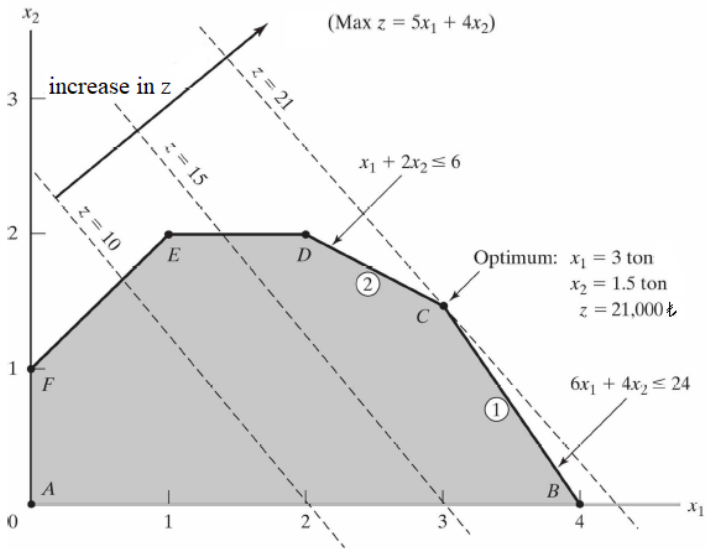
Optimum

$6x_1 + 4x_2 \leq 24$
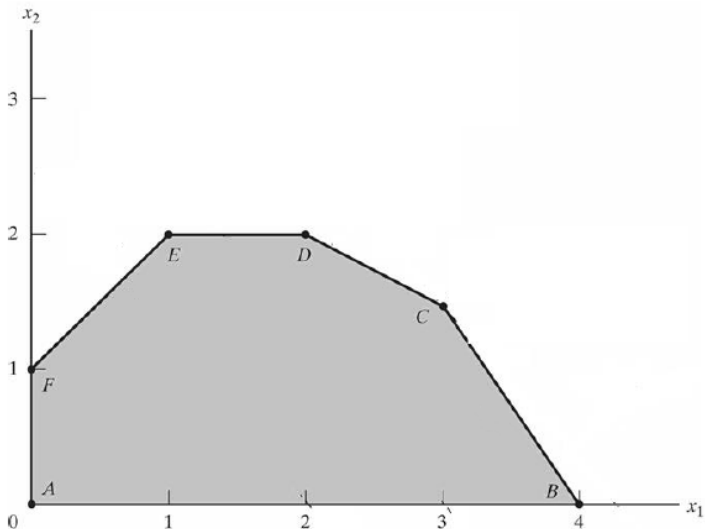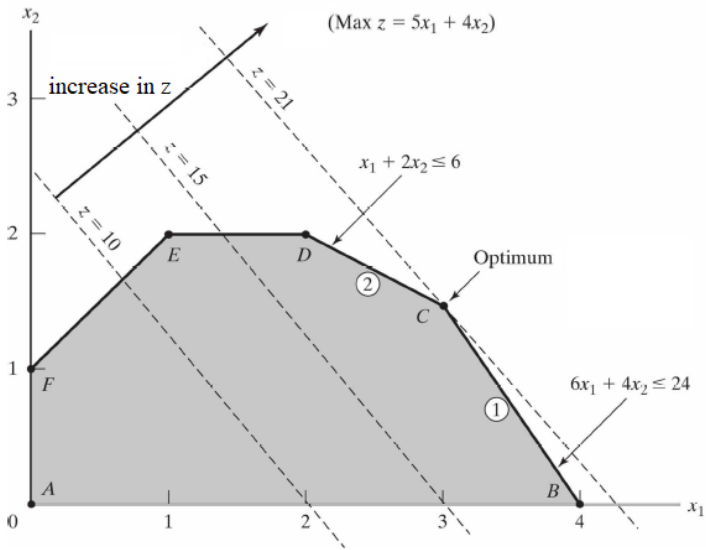
- **Step 2 (Determining the Optimum Solution-Continued):**
  It can be seen from the figure that the solution space exits at
  point C. Therefore, the optimum solution will occur at   .

  The point C is the intersection point of the lines shown with
  numbers 1 and 2 in the figure, i.e.

  The solution of these equations of these lines will give $x_1 =$
  3 and $x_2 = 1.5$. The objective function is calculated as
  $z = 5 \times 3 + 4 \times 1.5 = 21$. So, if 3 tons of exterior paint and
  1.5 tons of interior paint are produced in daily production,
  the associated daily profit will be 21000 TL.

It is not a coincidence that the optimum solution is at a corner point of the solution space. This is valid even when the objective function z is parallel to any constraint. For example, if the objective function was $z = 6x_1 + 4x_2$ (parallel to the constraint expressed by 1), corner points B or C would be found as the optimum solution. In fact, any point of the line segment BC would be an *alternative optimum* (see: (Taha, 2010, Example 3.5.2)).

This observation indicates that the LP optimum is always at the corner points and therefore the optimum solution can be found by examining the objective function values at the corner points of the region obtained from the constraints.

Corner Point	(x_1, x_2)	z
A	(0, 0)	
B	(4, 0)	
C	(3, 1.5)	
D	(2, 2)	
E	(1, 2)	
F	(0, 1)	

- **Method 1: Enumerating all the corner points**
  - Determine the feasible space.
  - List all the corner points.
  - By evaluating the z values at the corner points, select the 'best' as optimum.
- **Method 2: Determining the increasing/decreasing directions of z**
  - By assigning two arbitrary values to z, plot the iso-profit/iso-cost lines corresponding to these values.
  - Determine the direction.
  - Lines are drawn in the specified direction, parallel to the drawn z lines.
  - The last z line that intersects with the solution space is determined and this z line indicates the best z value, that is, the optimal solution of the LP.

- **Method 3: Gradient method:** As we know from analysis courses, moving in the direction of the gradient ($\vec{c}$) increases the z values. Similarly, when moving in the opposite direction of the gradient ($-\vec{c}$) objective function values decrease.

  That is why, in maximization problems, iso-profit lines are drawn in the direction of the gradient ($\vec{c}$) whereas in minimization problems, iso-cost lines are drawn in the opposite direction of the gradient ($-\vec{c}$). Finally, the last point obtained from the intersection of z lines with the solution space is determined as the optimum solution.

Example (Diet Problem)

At least 800 kg of silage (a special type of feed) is used per day in Ozark Farms. This special type of feed is obtained from a mixture of corn and soybean meal with the following compositions:

	kg per 1 kg of feedstuff		Cost (TL/kg)
	Protein	Fiber	
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

The composition of this special feed must contain at least 30 % protein and at most 5 % fiber. Ozark company wants to determine the minimum cost of the daily feed mix.

Since the feed consists of corn and soybean meal, the decision variables in the model are defined as follows:

The objective function here seeks the minimum of the total daily feed cost (in TL):

The constraints of the model should reflect both the amount of daily requirement and the characteristics of the mixture. Considering that Ozark Farms needs at least 800 kg of feed per day, we can express this constraint as follows:

Restrictions on protein diet requirements are also given below: The amount of protein in x_1 kg corn and x_2 kg soybean meal is $0.09x_1 + 0.6x_2$ kg. Since this amount is desired to be at least 30 % of the total product mix, we have

Similarly, the amount of fiber will be

When we rearrange all the constraints, the model with the objective function can be expressed as follows:

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