MTM3691-Theory of Linear Programming

Gökhan Göksu, PhD

Week 3

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The Simplex Method: LP Model in Equation Form

The problem must be standardized before basic solutions can be used when solving the general LP model. For a model to become standard, it must provide the following features:

1) All constraints, except those that require the variables to be non-negative, must be transformed into equations with a non-negative right hand side.
2) All variables must be zero or positive (not negative).
3) The objective function can be maximum or minimum.

Warning: Commercial softwares that uses the simplex method (e.g. TORA, LINDO, etc.) generally accepts inputs conforming to this standard form. In this context, the operating instructions should be read carefully before using these programs.

The Simplex Method: LP Model in Equation Form

1) **Turning inequalities into equalities:** An inequality of the form $\leq$ (or $\geq$) can be converted to an equality by adding a slack variable to the left hand side:
   - **For a constraint of the form $\leq$:**

     $$x_1 + 2x_2 \leq 3 \Leftrightarrow$$

     Here $s_1 \geq 0$ and it is called a **slack variable**.
   - **For a constraint of the form $\geq$:**

     $$3x_1 + x_2 \geq 5 \Leftrightarrow$$

     Here $S_1 \geq 0$ and it is called a **surplus variable**.

   The right-hand side of an equation must satisfy the non-negative condition. In fact, if necessary, the equation can be multiplied by -1 to achieve this. Another point that should be noted here is that an inequality of the form ($\leq$) can be transformed into an inequality of the form ($\geq$) by multiplying both sides by -1.

The Simplex Method: LP Model in Equation Form

2) **Converting unbounded variables into non-negative variables:** An unbounded variable x_j can be expressed in terms of two non-negative variables as shown below:

$$x_j = x_j^+ - x_j^-, \quad x_j^+, x_j^- \geq 0.$$

Warning: The process of substituting one variable for another as shown above will also affect all constraints and objective function. Once the problem is solved with variables x_j^+ and x_j^-, the value of the original variable must be determined by the operation $x_j = x_j^+ - x_j^-$.

3) **Converting maximization to minimization:** Maximization of a function $f(x_1, x_2, ..., x_n)$ is equivalent to minimization of the function $-f(x_1, x_2, ..., x_n)$. Both problems have the same optimal values of x_1, x_2, ..., x_n.

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

Graphical Method	Algebraic Method
Graph all constraints, including nonnegativity restrictions	Represent the solution space by m equations in n variables and restrict all variables to nonnegative values, $m < n$
Solution space consists of infinity of **feasible** points	The system has infinity of **feasible** solutions
↓	↓
Identify **feasible corner points** of the solution space	Determine the **feasible basic solutions** of the equations
Candidates for the optimum solution are given by a *finite* number of **corner points**	Candidates for the optimum solution are given by a *finite* number of **basic feasible solutions**
↓	↓
Use the objective function to determine the **optimum corner point** from among all the candidates	Use the objective function to determine the **optimum basic feasible solution** from among all the candidates

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

In the graphical method, since the solution space is divided into half-spaces determined by inequality constraints, there are an infinite number of solution points. In the algebraic method, since the solution space is expressed by m simultaneous linear equations containing n variables, if $m < n$, there is still an infinite number of solution points.

In this case ($m < n$); $n - m$ variables will take the value of zero, and the value of m variables will be determined by solving m equations.

If these m variables give *a unique solution*, the variables that satisfy this situation are called **basic variables**, and $n - m$ zero-valued variables are called **non-basic variables**.

In this case, the situation that results in a singular solution also includes **basic solution**. If we assume that all variables take non-negative values, then the basic solution is **feasible**. Otherwise, it is **infeasible**.

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

Based on the given definitions, the maximum number of basic solutions (or vertices) for m equations with n unknowns will be evaluated as follows?

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

Let us explain this with the following example.

Example

Let us consider the following two variable LP:

$$\begin{aligned} \max z = 2x_1 + 3x_2 & \\ \textit{subject to } 2x_1 + x_2 \leq & \ 4 \\ x_1 + 2x_2 \leq & \ 5 \\ x_1, x_2 \geq & \ 0 \end{aligned}$$

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Algebraically, the solution space of this LP is as follows.

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

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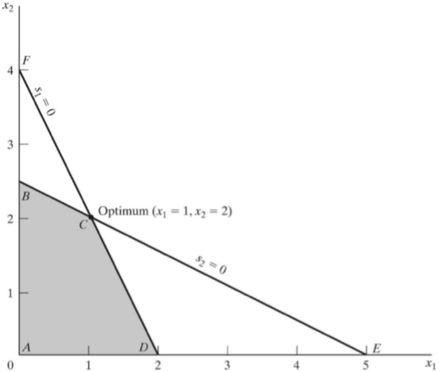

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

Now, for the system below, let us calculate the combinations of the basic feasible solution, the basic infeasible solution and the non-basic solution variables.

Number of all possible basic solutions (vertices)

$$\binom{4}{2} = \frac{4!}{2!2!} = 6.$$

Here,

- $n - m = 4 - 2 = 2$ is the number of variables that will take the value zero,
- $m = 2$ indicates the number of variables whose value will be determined by solving the equation.

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

Non-Basic Variables	Basic Variables	Basic Solution	Corner Point	Feasible?	Objective Function Value
(x_1, x_2)					
(x_1, s_1)					
(x_1, s_2)					
(x_2, s_1)	(x_1, s_2)	$(2, 3)$	D	Yes	
(x_2, s_2)	(x_1, s_1)	$(5, -6)$	E	No	
(s_1, s_2)	(x_1, x_2)	$(1, 2)$	C	Yes	

The Simplex Method (Summary)

- **Step 1:** By adding slack/surplus variables, inequalities are converted to equality form.
- **Step 2:** For m equations and n variables; by determining $n - m$ non-basic (zero) variables, a solution is found for the remaining m basic variables. If the basic solution is feasible, the objective function value is calculated.
- **Step 3:** For all basic variable combinations **Step 2** is applied.
- **Step 4:** The optimum solution is determined according to the maximum/minimum values of the objective function.