Lecture 5
Special Cases in the Simplex Method

Degeneracy
Alternative optima
Unbounded solutions
Nonexisting (or infeasible) solutions

3.5 SPECIAL CASES IN THE SIMPLEX METHOD

This section considers four special cases that arise in the use of the simplex method.

1. Degeneracy
2. Alternative optima
3. Unbounded solutions
4. Nonexisting (or infeasible) solutions

Our interest in studying these special cases is twofold: (1) to present a *theoretical* explanation of these situations and (2) to provide a *practical* interpretation of what these special results could mean in a real-life problem.

3.5.1 Degeneracy

In the application of the feasibility condition of the simplex method, a tie for the minimum ratio may occur and can be broken arbitrarily. When this happens, at least one *basic* variable will be zero in the next iteration and the new solution is said to be **degenerate**.

There is nothing alarming about a degenerate solution, with the exception of a small theoretical inconvenience, called **cycling** or **circling**, which we shall discuss shortly. From the practical standpoint, the condition reveals that the model has at least one *redundant* constraint. To provide more insight into the practical and theoretical impacts of degeneracy, a numeric example is used.

Example 3.5-1 (Degenerate Optimal Solution)

$$\text{Maximize } z = 3x_1 + 9x_2$$

subject to

$$x_1 + 4x_2 \le 8$$
$$x_1 + 2x_2 \le 4$$
$$x_1, x_2 \ge 0$$

Given the slack variables x_3 and x_4, the following tableaus provide the simplex iterations of the problem:

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-3	-9	0	0	0
x_2 enters	x_3	1	4	1	0	8
x_3 leaves	x_4	1	2	0	1	4
1	z	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18
x_1 enters	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0
2	z	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18
(optimum)	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2
	x_1	1	0	-1	2	0

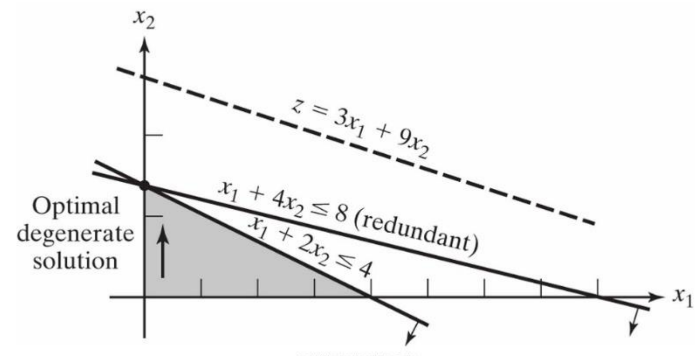


Figure 3.7
LP degeneracy in Example 3.5-1

PROBLEM SET 3.5A

2. Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2$$

subject to

$$4x_1 - x_2 \le 8$$
$$4x_1 + 3x_2 \le 12$$
$$4x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0$$

   (a) Show that the associated simplex iterations are temporarily degenerate (you may use TORA for convenience).
   (b) Verify the result by solving the problem graphically (TORA's Graphic module can be used here).

3.5.2 Alternative Optima

When the objective function is parallel to a nonredundant **binding constraint** (i.e., a constraint that is satisfied as an equation at the optimal solution), the objective function can assume the same optimal value at more than one solution point, thus giving rise to alternative optima. The next example shows that there is an *infinite* number of such solutions. It also demonstrates the practical significance of encountering such solutions.

Example 3.5-2 (Infinite Number of Solutions)

$$\text{Maximize } z = 2x_1 + 4x_2$$

subject to

$$x_1 + 2x_2 \leq 5$$

$$x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Figure 3.9 demonstrates how alternative optima can arise in the LP model when the objective function is parallel to a binding constraint. Any point on the *line segment BC* represents an alternative optimum with the same objective value $z = 10$.

The iterations of the model are given by the following tableaus.

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-2	-4	0	0	0
x_2 enters	x_3	1	2	1	0	5
x_3 leaves	x_4	1	1	0	1	4
1 (optimum)	z	0	0	2	0	10
x_1 enters	x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$
2	z	0	0	2	0	10
(alternative optimum)	x_2	0	1	1	-1	1
	x_1	1	0	-1	2	3

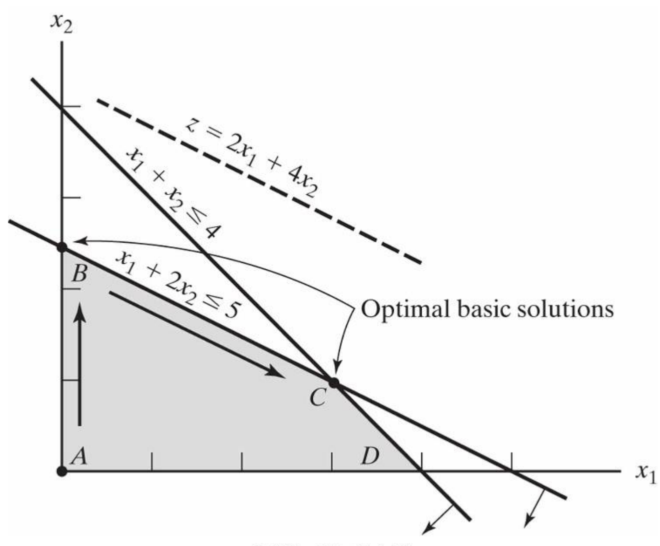


Figure 3.9
LP alternative optima in Example 3.5-2

Iteration 1 gives the optimum solution $x_1 = 0, x_2 = \frac{5}{2}$, and $z = 10$, which coincides with point B in Figure 3.9. How do we know from this tableau that alternative optima exist? Look at the z-equation coefficients of the *non*basic variables in iteration 1. The coefficient of nonbasic x_1 is zero, indicating that x_1 can enter the basic solution without changing the value of z, but causing a change in the values of the variables. Iteration 2 does just that—letting x_1 enter the basic solution and forcing x_4 to leave. The new solution point occurs at $C(x_1 = 3, x_2 = 1, z = 10)$. (TORA's Iterations option allows determining one alternative optimum at a time.)

The simplex method determines only the two corner points B and C. Mathematically, we can determine all the points (x_1, x_2) on the line segment BC as a nonnegative weighted average of points B and C. Thus, given

$$B: x_1 = 0, x_2 = \frac{5}{2}$$

$$C: x_1 = 3, x_2 = 1$$

then all the points on the line segment BC are given by

$$\left.\begin{aligned} \hat{x}_1 &= \alpha(0) + (1-\alpha)(3) = 3 - 3\alpha \\ \hat{x}_2 &= \alpha\left(\tfrac{5}{2}\right) + (1-\alpha)(1) = 1 + \tfrac{3}{2}\alpha \end{aligned}\right\}, 0 \le \alpha \le 1$$

When $\alpha = 0$, $(\hat{x}_1, \hat{x}_2) = (3,1)$, which is point C. When $\alpha = 1$, $(\hat{x}_1, \hat{x}_2) = \left(0, \frac{5}{2}\right)$, which is point B. For values of α between 0 and 1, $(\hat{x}_1, \hat{x}_2)$ lies between B and C.

Remarks. In practice, alternative optima are useful because we can choose from many solutions without experiencing deterioration in the objective value. For instance, in the present example, the solution at *B* shows that activity 2 only is at a positive level, whereas at *C* both activities are positive. If the example represents a product-mix situation, there may be advantages in producing two products rather than one to meet market competition. In this case, the solution at *C* may be more appealing.

PROBLEM SET 3.5B

*1. For the following LP, identify three alternative optimal basic solutions, and then write a general expression for all the nonbasic alternative optima comprising these three basic solutions.

$$\text{Maximize } z = x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 10 \\ x_1 + x_2 &\leq 5 \\ x_1 &\leq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Note: Although the problem has more than three alternative basic solution optima, you are only required to identify three of them. You may use TORA for convenience.

2. Solve the following LP:

$$\text{Maximize } z = 2x_1 - x_2 + 3x_3$$

subject to

$$x_1 - x_2 + 5x_3 \le 10$$
$$2x_1 - x_2 + 3x_3 \le 40$$
$$x_1, x_2, x_3 \ge 0$$

From the optimal tableau, show that all the alternative optima are not corner points (i.e., nonbasic). Give a two-dimensional graphical demonstration of the type of solution space and objective function that will produce this result. (You may use TORA for convenience.)

3. For the following LP, show that the optimal solution is degenerate and that none of the alternative solutions are corner points (you may use TORA for convenience).

$$\text{Maximize } z = 3x_1 + x_2$$

subject to

$$x_1 + 2x_2 \le 5$$
$$x_1 + x_2 - x_3 \le 2$$
$$7x_1 + 3x_2 - 5x_3 \le 20$$
$$x_1, x_2, x_3 \ge 0$$

3.5.3 Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraints—meaning that the solution space is *unbounded* in at least one variable. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded.

Unboundedness points to the possibility that the model is poorly constructed. The most likely irregularity in such models is that one or more nonredundant constraints have not been accounted for, and the parameters (constants) of some constraints may not have been estimated correctly.

The following examples show how unboundedness, in both the solution space and the objective value, can be recognized in the simplex tableau.

Example 3.5-3 (Unbounded Objective Value)

$$\text{Maximize } z = 2x_1 + x_2$$

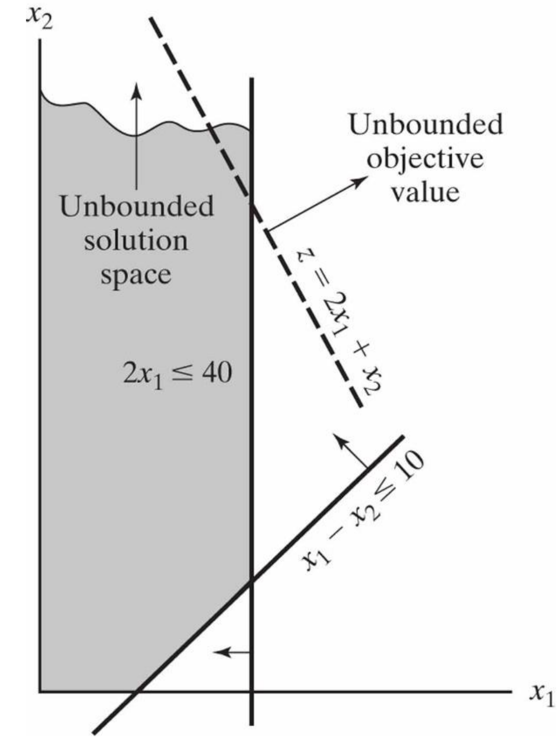
$$\text{subject to } \quad x_1 - x_2 \le 10$$

$$2x_1 \le 40$$

$$x_1, x_2 \ge 0$$

Starting Iteration

Basic	x_1	x_2	x_3	x_4	Solution
z	-2	-1	0	0	0
x_3	1	-1	1	0	10
x_4	2	0	0	1	40



In the starting tableau, both x_1 and x_2 have negative z-equation coefficients. Hence either one can improve the solution. Because x_1 has the most negative coefficient, it is normally selected as the entering variable. However, *all* the *constraint* coefficients under x_2 (i.e., the denominators of the ratios of the feasibility condition) are *negative* or *zero*. This means that there is no leaving variable and that x_2 can be increased indefinitely without violating any of the constraints (compare with the graphical interpretation of the minimum ratio in Figure 3.5). Because each unit increase in x_2 will increase z by 1, an infinite increase in x_2 leads to an infinite increase in z. Thus, the problem has no bounded solution. This result can be seen in Figure 3.10. The solution space is unbounded in the direction of x_2, and the value of z can be increased indefinitely.

PROBLEM SET 3.5C

1. *TORA Experiment.* Solve Example 3.5-3 using TORA's Iterations option and show that even though the solution starts with x_1 as the entering variable (per the optimality condition), the simplex algorithm will point eventually to an unbounded solution.

Basic	x_1	x_2	s_1	s_2	
z	-2	-1	0	0	0
s_1	1	-1	1	0	10
s_2	2	0	0	1	40
z	0	-3	2	0	20
x_1	1	-1	1	0	10
s_2	0	2	-2	1	20
z	0	0	-1	3/2	50
x_1	1	0	0	1/2	20
x_2	0	1	-1	1/2	10

unbounded → (s_1 column)

*2. Consider the LP:

$$\text{Maximize } z = 20x_1 + 10x_2 + x_3$$

subject to

$$\begin{aligned} 3x_1 - 3x_2 + 5x_3 &\le 50 \\ x_1 \qquad\quad + x_3 &\le 10 \\ x_1 - x_2 + 4x_3 &\le 20 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

(a) By inspecting the constraints, determine the direction (x_1, x_2, or x_3) in which the solution space is unbounded.

(b) Without further computations, what can you conclude regarding the optimum objective value?

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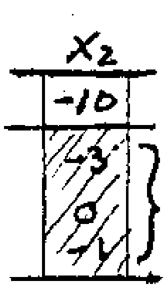
(a) 

$\Rightarrow$ Solution space unbounded in the direction of x_2

(b) Objective value is unbounded because each unit increase in x_2 increases z by 10

3.5.4 Infeasible Solution

LP models with inconsistent constraints have no feasible solution. This situation can never occur if *all* the constraints are of the type $\leq$ with nonnegative right-hand sides because the slacks provide a feasible solution. For other types of constraints, we use artificial variables. Although the artificial variables are penalized in the objective function to force them to zero at the optimum, this can occur only if the model has a feasible space. Otherwise, at least one artificial variable will be *positive* in the optimum iteration. From the practical standpoint, an infeasible space points to the possibility that the model is not formulated correctly.

Example 3.5-4 (Infeasible Solution Space)

Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Using the penalty $M = 100$ for the artificial variable R, the following tableaux provide the simplex iterations of the model.

Iteration	Basic	x_1	x_2	x_4	x_3	R	Solution
0	z	−303	−402	100	0	0	−1200
x_2 enters	x_3	2	1	0	1	0	2
x_3 leaves	R	3	4	−1	0	1	12
1	z	501	0	100	402	0	−396
(pseudo-optimum)	x_2	2	1	0	1	0	2
	R	−5	0	−1	−4	1	4

Optimum iteration 1 shows that the artificial variable R is *positive* (= 4), which indicates that the problem is infeasible. Figure 3.11 demonstrates the infeasible solution space. By allowing the artificial variable to be positive, the simplex method, in essence, has reversed the direction of the inequality from $3x_1 + 4x_2 \geq 12$ to $3x_1 + 4x_2 \leq 12$ (can you explain how?). The result is what we may call a **pseudo-optimal** solution.

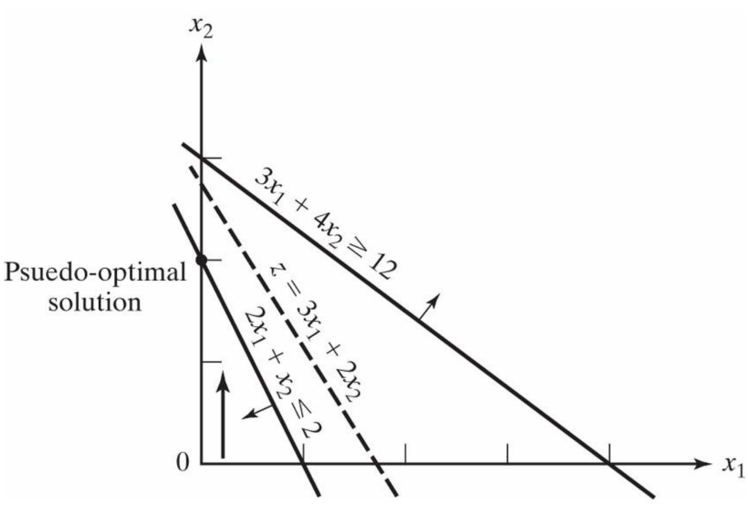


Figure 3.11
Infeasible solution of Example 3.5-4

PROBLEM SET 3.5D

*1. Toolco produces three types of tools, $T1$, $T2$, and $T3$. The tools use two raw materials, $M1$ and $M2$, according to the data in the following table:

	Number of units of raw materials per tool		
Raw material	$T1$	$T2$	$T3$
$M1$	3	5	6
$M2$	5	3	4

The available daily quantities of raw materials $M1$ and $M2$ are 1000 units and 1200 units, respectively. The marketing department informed the production manager that according to their research, the daily demand for all three tools must be at least 500 units. Will the manufacturing department be able to satisfy the demand? If not, what is the most Toolco can provide of the three tools?

x_1 = number of units of T1
x_2 = number of units of T2
x_3 = number of units of T3

Constraints:

$$3x_1 + 5x_2 + 6x_3 \le 1000$$
$$5x_1 + 3x_2 + 4x_3 \le 1200$$
$$x_1 + x_2 + x_3 \ge 500$$
$$x_1, x_2, x_3 \ge 0$$

$$3x_1 + 5x_2 + 6x_3 + S_1 = 1000$$
$$5x_1 + 3x_2 + 4x_3 + S_2 = 1200$$
$$x_1 + x_2 + x_3 - S_3 + R_3 = 500$$
$$x_1, x_2, x_3, S_1, S_2, S_3, R_3 \ge 0$$

Optimum solution from TORA

$R_3 = r = 225$ units

This is interpreted as a deficiency of 225 units. The most that can be produced is $500 - 225 = 275$ units

2. *TORA Experiment.* Consider the LP model

$$\text{Maximize } z = 3x_1 + 2x_1 + 3x_3$$

subject to

$$2x_1 + x_2 + x_3 \le 2$$
$$3x_1 + 4x_2 + 2x_3 \ge 8$$
$$x_1, x_2, x_3 \ge 0$$

Use TORA's Iterations ⇒ M-Method to show that the optimal solution includes an artificial basic variable, but at zero level. Does the problem have a *feasible* optimal solution?

Basic	x_1	x_2	x_3	s_1	s_2	R_1	Sol^n
z	$-3-3M$	$-2-4M$	$-3-2M$	M	0	0	$-8M$
s_1	2	[1]	1	0	1	0	2
R_1	3	4	2	−1	0	1	8
z	$-1+5M$	0	$-1+2M$	M	$2+4M$	0	4
x_2	2	1	1	0	1	0	2
R_1	−5	0	−2	−1	−4	1	0

Because $R_1 = 0$ in the optimal tableau, the problem has a feasible solution. The optimum solution is

$x_1 = 0,\ x_2 = 2,\ z = 4$

Note that in the first iteration, R_1 could have been used as the leaving variable, in which case it would not be basic in the optimum iteration.

Reference: *H.A.Taha, Operations Research: An Introduction,* Prentice Hall; 9th edition, *Singapore, 2010.*