MTM3691-Theory of Linear Programming

Gökhan Göksu, PhD

Week 10

Course Content

- Chapter 2: Introduction to Linear Programming (LP)
- Chapter 3: Simplex Method
- Chapter 4: Duality and Sensitivity Analysis
  - Graphical Sensitivity Analysis
  - Algebraic Sensitivity Analysis
- Chapter 5: Transportation Model and Various Transportation Models

Graphical Sensitivity Analysis

In linear programming, model parameters (inputs) can vary within certain limits without affecting the optimal solution. This situation is referred to as **sensitivity analysis**. Later, we will explore **post-optimality analyses** related to determining the new optimal solution due to targeted changes in input data. Let's consider the following example for graphical sensitivity analysis.

Example

Göksu Inc. produces two types of products on two machines. For product 1, it takes 2 hours on machine 1 and 1 hour on machine 2. For product 2, it takes 1 hour on machine 1 and 3 hours on machine 2. The unit revenues for products 1 and 2 are 30 TL and 20 TL, respectively. The total available daily processing time for each machine is 8 hours.

Let x_1 and x_2 represent the daily production quantities for products 1 and 2, respectively. The LP problem can be written as follows:

$$\max z = 30x_1 + 20x_2$$

$$\begin{aligned} \text{subject to:} \quad & 2x_1 + x_2 \leq 8 \quad &\text{(Machine 1)} \\ & x_1 + 3x_2 \leq 8 \quad &\text{(Machine 2)} \\ & x_1, x_2 \geq 0 \end{aligned}$$

Graphical Sensitivity Analysis

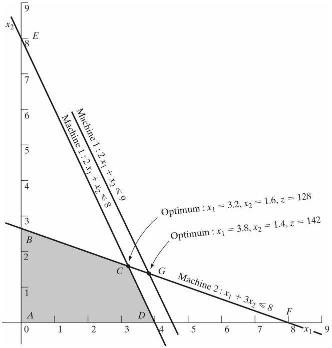

Figure: Sensitivity of the optimal solution to changes in resource availability (right-hand side of constraints)

$$\begin{pmatrix} \text{Increase in the capacity of} \\ \text{Machine 1 by 1 hour} \\ \text{(from point C to point G)} \end{pmatrix} =$$

Graphical Sensitivity Analysis

The calculated ratio provides a **unit value of a resource** (in TL/hour) representing a **direct link** between a model input (resources) and its output (total revenue). It shows the sensitivity of the change in the optimal objective value per unit change in the input (machine capacity). Numerically, it indicates that a unit increase (decrease) in the capacity of Machine 1 will increase (decrease) the revenue by 14 TL/hour. While the unit value of a resource is an appropriate definition for the rate of change of the objective function, it is referred to as **dual** or **shadow price** in LP literature and all software packages.

Given the dual price's applicability range:

$$\text{Minimum Machine 1 capacity [B=(0,2.67)]} =$$

$$\text{Maximum Machine 1 capacity [F=(8,0)]} =$$

Thus, it can be concluded that the dual price of 14 TL/hour is valid for this range:

$$2.67 \text{ hours} \leq \text{Machine 1 capacity} \leq 16 \text{ hours}$$

Changes outside this range produce different dual prices (unit values).

Graphical Sensitivity Analysis

Similar calculations can determine that the dual price for Machine 2 capacity is 2 TL/hour, valid for any changes (increases or decreases) moving along the DE segment, parallel to itself. This leads to the following limits:

$$\text{Minimum Machine 2 capacity [D=(4,0)]} = 1 \times 4 + 3 \times 0 = 4 \text{ hours}$$
$$\text{Maximum Machine 2 capacity [E=(0,8)]} = 1 \times 0 + 3 \times 8 = 24 \text{ hours}$$

It can be concluded that the dual price of 2 TL/hour is applicable for this range for Machine 2:

$$4 \text{ hours} \leq \text{Machine 2 capacity} \leq 24 \text{ hours}$$

The calculated ranges for Machines 1 and 2 are referred to as the **feasibility range**.

Graphical Sensitivity Analysis

Dual prices play a role in decision-making for the economic model represented by the relevant LP problem. Returning to our example, the decision-making stage raises the following questions:

Question

If Göksu Inc. can increase the capacities of both machines, which machine should be prioritized?

The dual prices for Machines 1 and 2 are 14 TL/hour and 2 TL/hour, respectively. These prices mean that an additional hour of Machine 1 will increase the revenue by 14 TL, while Machine 2 will only increase it by 2 TL. Therefore, priority should be given to Machine 1.

Question

A proposal has been made to increase the capacities of Machines 1 and 2 by 10 TL per hour. What recommendation should be made in this regard?

The net additional revenue per hour for Machine 1 is $14 - 10 = 4$ TL, and for Machine 2, it is $2 - 10 = -8$ TL. Therefore, only the capacity of Machine 1 should be increased.

Question

If the capacity of Machine 1 is increased from 8 hours to 13 hours, how does this increase affect the optimal revenue?

The dual price for Machine 1 is 14 TL, and it is in the range $(2.67, 16)$ hours. The proposed increase to 13 hours is within the feasibility range. Thus, the increase in revenue is $14(13 - 8) = 70$ TL, resulting in a total revenue of $(\text{current value} + \text{increase in revenue}) = 128 + 70 = 198$ TL.

Question

If the capacity of Machine 1 is increased to 20 hours, how does this increase affect the optimal revenue?

The proposed change is outside the range $(2.67, 16)$ hours, where the dual price of 14 TL/hour is valid. Therefore, a conclusion can only be reached for an immediate increase of up to 16 hours. Beyond that, further calculations are needed to determine the solution, indicating that we don't have enough information to make an immediate decision.

Question

It is known that as long as a change in a resource is within the feasibility range, the change in the optimal objective value is equal to the "dual price $\times$ change in the resource." What can be said about the associated optimal values of the variables?

The optimal values of the variables will definitely change. However, the level of information obtained from graphical sensitivity analysis is not sufficient to determine the new values. For this purpose, algebraic sensitivity analysis must be performed.

Example

The graph below illustrates the graphical solution of the previous example for Göksu Inc. The optimal solution is at point C ($x_1 = 3.2$, $x_2 = 1.6$, $z = 128$). Changes in revenue (i.e., changes in the coefficients of the objective function) will alter the slope of z. However, as can be seen from the figure, as long as the objective function lies between the BF and DE lines, defining the optimal point, the solution will remain at point C. This situation implies that there is a range for the coefficients of the objective function that will keep the optimal solution at C without changing.

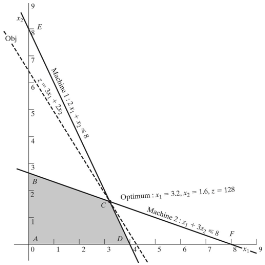

Figure: Graphical sensitivity of the optimal solution to changes in revenue (coefficients of the objective function).

Graphical Sensitivity Analysis

The objective function can be written in general form as:

$$\max \; z = c_1x_1 + c_2x_2$$

For the scenario where the z line can rotate both clockwise and counterclockwise at point C, maintaining its position between the lines $z = c_1x_1 + c_2x_2$, $x_1 + 3x_2 = 8$, and $2x_1 + x_2 = 8$, the ratio $\frac{c_1}{c_2}$ can vary between $\frac{1}{3}$ and $\frac{2}{1}$, yielding the following condition:

This information allows us to answer some questions about the optimal solution.

Question

Assuming that the revenues for products 1 and 2 are changed to 35 TL and 25 TL, respectively, can the current optimum remain the same?

The new objective function will be:

$$\max \ z = 35x_1 + 25x_2$$

The solution at point C will remain optimal because $\frac{c_1}{c_2} = \frac{35}{25} = 1.4$ stays within the optimality range, i.e., $(0.3333, 2)$. This ratio will remain within this range even if it falls outside. This means there is an interval that will keep the optimal solution at C without changing, as long as the ratio stays between $\frac{1}{3}$ and $\frac{2}{1}$.

Question

For a given $c_2 = 20$ TL, if the c_1 for the current optimum remains the same, what is the associated range?

Substituting $c_2 = 20$ into $\frac{1}{3} \leq \frac{c_1}{c_2} \leq 2$ yields:

This range is referred to as the **optimality range** for c_1, and it assumes c_2 is fixed at 20 TL.

Similarly, the optimality range for c_2 can be obtained by fixing c_1 at 30 TL:

$$c_2 \leq 30 \times 3 \text{ and } c_2 \geq \frac{30}{2} \implies 15 \leq c_2 \leq 90$$

Example

Goksu Inc. produces three types of toys using three operations: train, truck, and car. The daily time limits for the three operations are 430, 460, and 420 minutes, respectively. The income per toy for train, truck, and car is 3 TL, 2 TL, and 5 TL, respectively. The assembly times per toy for the three operations are 1, 3, and 1 minute, respectively. The corresponding times per train and car are (2, 0, 4) and (1, 2, 0) minutes. Zero time indicates that the operation is not used.

Let x_1, x_2, and x_3 represent the daily production quantities of train, truck, and car, respectively. The LP model is given as follows:

$$\begin{aligned} \max z &= 3x_1 + 2x_2 + 5x_3 \\ \text{Subject to:} \quad x_1 + 2x_2 + x_3 &\leq 430 \quad \text{(Operation 1)} \\ 3x_1 + 2x_3 &\leq 460 \quad \text{(Operation 2)} \\ x_1 + 4x_2 &\leq 420 \quad \text{(Operation 3)} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Algebraic Sensitivity Analysis

Let x_4, x_5, and x_6 represent the slack variables for operations 1, 2, and 3, respectively. The optimal tableau is given as follows:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	-1/4	1	0	1/2	-1/4	0	100
x_3	3/2	0	1	0	1/2	0	230
x_6	2	0	0	-2	1	1	20

Determination of Dual Prices: After introducing slack variables x_4, x_5, and x_6, the model constraints can be written as:

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 430 \quad \text{(Operation 1)} \\ 3x_1 + 2x_3 + x_5 &= 460 \quad \text{(Operation 2)} \\ x_1 + 4x_2 + x_6 &= 420 \quad \text{(Operation 3)} \end{aligned}$$

Subtracting slack variables from both sides, we get:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 430 - x_4 \quad \text{(Operation 1)} \\ 3x_1 + 2x_3 &= 460 - x_5 \quad \text{(Operation 2)} \\ x_1 + 4x_2 &= 420 - x_6 \quad \text{(Operation 3)} \end{aligned}$$

Algebraic Sensitivity Analysis

In this representation, slack variables are in the same units (minutes) as operation times. Therefore, a one-minute decrease in slack variables is equivalent to a one-minute increase in operation times. We can use this information to determine the dual prices from the objective function in the optimal tableau:

$$z + 4x_1 + x_4 + 2x_5 + 0x_6 = 1350.$$

Rewriting this equation:

A decrease in slack variable values results in an increase in the corresponding operation times. Therefore, this equation can be expressed in terms of operation times as:

This equation indicates that a one-minute increase in the time of Operation 1 results in a 1 TL increase in z, a one-minute increase in the time of Operation 2 results in a 2 TL increase in z, and a one-minute increase in the time of Operation 3 does not change z.

Algebraic Sensitivity Analysis

In summary, the impact of slack variables on dual prices is shown in the table below.

Resource	Slack Variable	Coefficient in the z-row of the optimal tableau	Dual Price
Operation 1	x_4	1	1 TL/minute
Operation 2	x_5	2	2 TL/minute
Operation 3	x_6	0	0 TL/minute

A zero dual price for Operation 3 implies that allocating more production time to this operation has no economic advantage. As evident from the positive (=20) value of the slack variable associated with Operation 3 in the optimal solution, it is meaningful since the resource is already abundant. For each of Operations 1 and 2, a one-minute increase will result in an increase in income by 1 TL and 2 TL, respectively. Additionally, dual prices indicate that, when allocating additional resources, Operation 2 might be given higher priority than Operation 1, as the dual price for Operation 2 is twice that of Operation 1.

Algebraic Sensitivity Analysis

Determining the Feasibility Range: Let D_1, D_2, and D_3 represent (positive or negative) changes in the daily production times allocated to Operations 1, 2, and 3, respectively. The LP problem containing these changes can be written as follows:

$$\max z = 3x_1 + 2x_2 + 5x_2$$

$$\begin{aligned} \text{Subject to:} \quad x_1 + 2x_2 + x_3 &\leq 430 + D_1 \quad \text{(Operation 1)} \\ 3x_1 + 2x_3 &\leq 460 + D_2 \quad \text{(Operation 2)} \\ x_1 + 4x_2 &\leq 420 + D_3 \quad \text{(Operation 3)} \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The relevant procedure is based on recalculating the modified right-hand side of the optimal one-way tableau and then deriving the conditions that would allow a solution. To demonstrate how the right-hand side is recalculated, the solution column of the starting tableau is replaced using the new right-hand sides:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS	D_1	D_2	D_3
z	-3	-2	-5	0	0	0	0	0	0	0
x_4	1	2	1	1	0	0	430	1	0	0
x_5	3	0	2	0	1	0	460	0	1	0
x_6	1	4	0	0	0	1	420	0	0	1

Algebraic Sensitivity Analysis

After solving the problem, the following tableau is obtained.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS	D_1	D_2	D_3
z	4	0	0	1	2	0	1350	1	2	0
x_2	-1/4	1	0	1/2	-1/4	0	100	1/2	-1/4	0
x_3	3/2	0	1	0	3/2	0	230	0	1/2	0
x_6	2	0	0	-2	1	1	20	-2	1	1

According to the new optimal tableau, the following optimal solution is obtained:

As shown earlier, the new z value confirms that the dual prices for Operations 1, 2, and 3 are 1, 2, and 0, respectively.

Algebraic Sensitivity Analysis

The solution found will remain appropriate as long as it satisfies the condition that all variables are non-negative:

$$x_2 = 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \geq 0$$
$$x_3 = 230 + \frac{1}{2}D_2 \geq 0$$
$$x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0$$

After simultaneous changes in D_1, D_2, and D_3, the solution will be appropriate as long as the inequalities are satisfied. Once all conditions are met, the new optimal solution can be found by directly substituting D_1, D_2, and D_3 into the equations provided above.

Algebraic Sensitivity Analysis

To demonstrate the use of these conditions, let's assume the current production times for Operations 1, 2, and 3 are 480, 440, and 410 minutes, respectively. $D_1 = 480 - 430 = 50$, $D_2 = 440 - 460 = -20$, and $D_3 = 410 - 420 = -10$. Substituting into the feasibility conditions:

$$
\begin{aligned}
x_2 =& 100 + \frac{1}{2}(50) - \frac{1}{4}(-20) = 130 > 0 && \text{(Feasible)} \\
x_3 =& 230 + \frac{1}{2}(-20) = 220 > 0 && \text{(Feasible)} \\
x_6 =& 20 - 2(50) + (-20) + (-10) = -110 < 0 && \text{(Infeasible)}
\end{aligned}
$$

Since the last condition $x_6 < 0$ is not satisfied according to the calculation, the current solution is no longer feasible. New calculations are required for a new solution, which will be explored under the heading "Post-Optimal Analysis."

Algebraic Sensitivity Analysis

Alternatively, if changes in resources are $D_1 = -30$, $D_2 = -12$, and $D_3 = 10$:

$$x_2 = 100 + \frac{1}{2}(-30) - \frac{1}{4}(-12) = 88 > 0 \qquad \text{(Feasible)}$$

$$x_3 = 230 + \frac{1}{2}(-12) = 224 > 0 \qquad \text{(Feasible)}$$

$$x_6 = 20 - 2(-30) + (-12) + (10) = 78 > 0 \qquad \text{(Feasible)}$$

The new feasible solution is $x_1 = 88$, $x_3 = 224$, $x_6 = 68$, and the corresponding objective function value is $z = 3(0) + 2(88) + 5(224) = 1296$ TL. The same new optimal objective function value can also be calculated as $z = 1350 + 1(-30) + 2(-12) = 1296$ TL.

Given conditions can be customized to calculate individual feasibility intervals resulting from changing each resource.

Algebraic Sensitivity Analysis

- **Scenario 1 (Change in Operation 1 Time from 460 to 460+D_1):** This change corresponds to selecting $D_2 = D_3 = 0$. This provides the following range:

- **Scenario 2 (Change in Operation 2 Time from 430 to 430+D_2):** This change corresponds to selecting $D_1 = D_3 = 0$. This provides the following range:

$$\left.\begin{array}{lll} x_2 = 100 - \frac{1}{4}D_2 \geq 0 & \Longrightarrow & D_2 \leq 400 \\ x_3 = 230 + \frac{1}{2} \geq 0 & \Longrightarrow & D_2 \geq -460 \\ x_6 = 20 + D_2 \geq 0 & \Longrightarrow & D_2 \leq -20 \end{array}\right\} \Longrightarrow -20 \leq D_2 \leq 400$$

Algebraic Sensitivity Analysis

- **Scenario 3 (Change in Operation 3 Time from 420 to 420+D_3):** This change corresponds to selecting $D_1 = D_2 = 0$. This provides the following range:

$$\left.\begin{aligned} x_2 &= 100 > 0 \\ x_3 &= 230 > 0 \\ x_6 &= 20 + D_3 \geq 0 \end{aligned}\right\} \implies -20 \leq D_3 < \infty$$

Dual prices and relevant feasibility intervals can be summarized as shown in the table below:

Source	Dual price	Feasibility interval	Minimum resource (min)	Current resource (min)	Maximum resource (min)
Op. 1	1	$-200 \leq D_1 \leq 10$	230	430	440
Op. 2	2	$-20 \leq D_2 \leq 400$	440	440	860
Op. 3	0	$-20 \leq D_3 < \infty$	400	420	∞

Algebraic Sensitivity Analysis

Even if changes violate some intervals, dual prices will remain valid for any simultaneous changes that maintain feasibility. For example, with $D_1 = 30$, $D_2 = -12$, and $D_3 = 100$, the solution will remain feasible, despite violating the feasibility interval $-200 \leq D_1 \leq 10$, as shown by the following calculations:

$$x_2 = 100 + \frac{1}{2}(30) - \frac{1}{4}(-12) = 118 > 0 \qquad \text{(Feasible)}$$

$$x_3 = 230 + \frac{1}{2}(-12) = 224 > 0 \qquad \text{(Feasible)}$$

$$x_6 = 20 - 2(30) + (-12) + (100) = 48 > 0 \qquad \text{(Feasible)}$$

In this case, it means that the dual prices will remain applicable, and the new optimal objective value can be calculated as $z = 1350 + 1(30) + 2(-12) + 0(100) = 1356$ TL.

Algebraic Sensitivity Analysis

The results examined can be summarized as follows:

- Changes in the right-hand side (D_i, $i = 1, 2, ..., m$) of the constraints will keep dual prices valid as long as feasibility is maintained simultaneously, or the changes remain within the feasibility intervals when made individually.
- In cases where simultaneous feasibility conditions are not met, or some feasibility intervals are violated, dual prices are not valid. In such cases, the approach is either to solve the problem with new D_i values or to apply post-optimal analysis.

Algebraic Sensitivity Analysis: Objective Function

Reduced Cost: For the sensitivity analysis of the objective function, we first need to define the **reduced costs**. In the Göksu Inc. model, the z-equation in the optimal table is as follows:

$$z + 4x_1 + x_4 + 2x_5 = 1350 \implies z = 1350 - 4x_1 - x_4 - 2x_5$$

In the optimal solution, it was suggested not to produce toy trains ($x_1 = 0$). This suggestion is also confirmed by the information in the z-equation because any unit increase above the current zero level in x_1 will decrease the value of z by 4 TL:

$$z = 1350 - 4 \times (1) - 1 \times (0) - 2 \times (0) = 1346 \text{ TL}.$$

We can consider the coefficient of x_1 in the z-equation (=4) as the unit cost because it leads to a decrease in z revenue. It is known in the original model that x_1 has a unit profit of 3 TL. Also, it is known that each toy train consumes resources (processing time), incurring a certain cost. Therefore, from an optimization perspective, the "preference" for x_1 depends on the unit values of revenue and the unit cost of consumed resources. This relationship has been expressed in the linear programming literature by defining the reduced cost as follows:

$$\begin{pmatrix}\text{Unit Reduced} \\ \text{Cost}\end{pmatrix} = \begin{pmatrix}\text{Unit cost of consumed} \\ \text{resources}\end{pmatrix} - (\text{Unit gain})$$

Algebraic Sensitivity Analysis: Objective Function

With the definition of **reduced cost**, an unprofitable variable (e.g., x_1) can be made profitable in two ways:

- By increasing unit profit,
- By reducing the unit cost of consumed resources.

Determining Optimality Ranges: Determining the conditions that will ensure the non-change of the optimal solution depends on the definition of reduced costs.

In the Göksu Inc. model, let d_1, d_2, and d_3 be the unit profit changes for toy trucks, trains, and cars, respectively. Thus, the objective function will be

As done for the right-hand side sensitivity analysis, the general case where all coefficients of the objective function change simultaneously will be considered first, followed by the case where the results are specialized for each scenario.

With simultaneous changes, the order of the z-row in the starting tableau will be as follows:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-3-d_1$	$-2-d_2$	$-5-d_3$	0	0	0	0

Algebraic Sensitivity Analysis: Objective Function

When constructing one-way tables using the same variable entry and exit order as in the original model (before applying d_j changes), the optimal iteration will be as follows:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$	0	0	$1 + \frac{1}{2}d_2$	$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3$	0	$1350 + 100d_2 + 230d_3$
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	100
x_6	$-\frac{1}{4}$	0	0	-2	1	1	20

The new optimal tableau is identical to the original optimal tableau, except for changes in the reduced costs (coefficients of the z-equation). This implies that changes in the coefficients of the objective function can only affect the optimality of the problem.

To calculate the new reduced costs, there's no need to perform row operations. Examining the new z-row shows that the coefficients of d_j are directly taken from the constraint coefficients of the optimal table. A convenient way to calculate the reduced costs is to add a new top row and a new leftmost column to the optimal table, as shown in the next slide, highlighted in red. The top row elements specify the changes associated with each variable (d_j's). The leftmost column elements are 1 in the z-row and the corresponding d_j in each basic variable row. It should be noted that $d_j \equiv 0$ for slack variables.

Algebraic Sensitivity Analysis: Objective Function

		d_1	d_2	d_3	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
1	z	4	0	0	1	2	0	1350
d_2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
d_3	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	100
0	x_6	2	0	0	-2	1	1	20

To compute the new reduced cost of any variable (or z), the elements in the relevant column are multiplied by the corresponding elements in the leftmost column, summed, and the result is subtracted from the top-row element. For example, for x_1, consider the table below:

	d_1	
Left Column	x_1	(Left Column $\times$ x_1 Column)
1	4	4×1
d_2	$-\frac{1}{4}$	$-\frac{1}{4}d_2$
d_3	$\frac{3}{2}$	$\frac{3}{2}d_3$
0	2	2×0
Reduced cost for $x_1 = 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$		

Algebraic Sensitivity Analysis: Objective Function

Applying these calculations to basic variables always results in zero reduced cost. Additionally, applying the same rule to the solution column produces $z = 1350 + 100d_2 + 230d_3$.

Since we are dealing with a maximization problem, the current solution remains optimal as long as the new reduced costs (coefficients in the z equation) for all non-basic variables are non-negative. Thus, the **optimality conditions** for the non-basic x_1, x_4, and x_5 are obtained:

$$\begin{aligned} 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 &\geq 0 \\ 1 + \frac{1}{2}d_2 &\geq 0 \\ 2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 &\geq 0. \end{aligned}$$

To maintain the current optimum's optimality, these conditions must be simultaneously satisfied. To demonstrate the use of these conditions, assume the objective function of the Göksu Inc. example is modified as follows:

$$\max\ z = 3x_1 + 2x_2 + 5x_3 \quad \rightarrow \quad \max\ z = 2x_1 + x_2 + 6x_3$$

Algebraic Sensitivity Analysis: Objective Function

Then, $d_1 = 2 - 3 = -1$ TL, $d_2 = 1 - 2 = -1$ TL, and $d_3 = 6 - 5 = 1$ TL. Substituting these values into the conditions:

$$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 = 4 - \frac{1}{4}(-1) + \frac{3}{2}(1) - (-1) = 6.75 > 0$$

$$1 + \frac{1}{2}d_2 = 1 + \frac{1}{2}(-1) = \frac{1}{2} > 0$$

$$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 = 2 - \frac{1}{4}(-1) + \frac{1}{2}(1) = \frac{11}{4} > 0.$$

The results indicate that the proposed changes will keep the current solution ($x_1 = 0$, $x_2 = 100$, $x_3 = 230$) optimal. Therefore, there is no need for further calculations, except for the change in the objective function value, $z = 1350 + 100d_2 + 230d_3 = 1350 + 100(-1) + 230(1) = 1480$ TL. New solution calculations should be made for cases where the relevant conditions are not satisfied.

So far, sensitivity analysis related to the maximization case has been discussed. In the minimization case, the only difference is that the reduced costs (z-equation coefficients) must not be negative to maintain optimality.

Algebraic Sensitivity Analysis: Objective Function

General optimality conditions can be used to determine a special case where changes are simultaneously made individually. This analysis is equivalent to considering the following three cases:

- $\max\ z = (3 + d_1)x_1 + 2x_2 + 5x_3$
- $\max\ z = 3x_1 + (2 + d_2)x_2 + 5x_3$
- $\max\ z = 3x_1 + 2x_2 + (5 + d_3)x_3$

These conditions can be explained individually as special cases of the simultaneous situation:

- **Case 1:** Taking $d_2 = d_3 = 0$ in simultaneous conditions:

- **Case 2:** Taking $d_1 = d_3 = 0$ in simultaneous conditions:

Algebraic Sensitivity Analysis: Objective Function

- **Case 3:** Taking $d_1 = d_2 = 0$ in simultaneous conditions:

$$\left.\begin{array}{lcl} 4 + \frac{3}{2}d_3 \geq 0 & \Longrightarrow & d_3 \geq -\frac{8}{3} \\ 2 + \frac{1}{2}d_3 \geq 0 & \Longrightarrow & d_3 \geq -4 \end{array}\right\} \Longrightarrow -\frac{8}{3} \leq d_3 < \infty$$

It should be noted that without satisfying simultaneous conditions, d_1, d_2, and d_3 changes can be within the allowed relevant intervals, and vice versa. For example,

$$\max\ z = 6x_1 + 8x_2 + 3x_3$$

Considered here, $d_1 = 6 - 3 = 3$ TL, $d_2 = 8 - 2 = 6$ TL, and $d_3 = 3 - 5 = -2$ TL. These values are individually within the desired intervals:

$$-\infty < d_1 \leq 4, \qquad -2 \leq d_2 \leq 8, \qquad -\frac{8}{3} \leq d_3 < \infty.$$

However, simultaneous conditions may not satisfy all inequalities, as shown below:

$$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 = 4 - \frac{1}{4}(6) + \frac{3}{2}(-2) - 3 = -\frac{7}{2} < 0$$

$$1 + \frac{1}{2}d_2 = 1 + \frac{1}{2}(6) = 4 > 0$$

$$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 = 2 - \frac{1}{4}(6) + \frac{1}{2}(-2) = -\frac{1}{2} < 0.$$

Algebraic Sensitivity Analysis: Objective Function

The examined results can be summarized as follows:

- Changes in d_j in the coefficients of the objective function, where $j = 1, 2, ..., n$, keep the optimal values of the variables unchanged when changes are simultaneous and satisfy all optimality conditions or individually stay within the optimality intervals.
- For other cases where simultaneous optimality conditions are not met or individual feasibility intervals are violated, the approach is to solve the problem with new d_j values or apply post-optimal analysis.

Example

The optimal table for the problem below is given. Suppose the coefficient of x_1 in the objective function is $9 + \lambda$. Determine the λ range to not violate the optimality condition.

$$\begin{aligned} \max z &= 9x_1 + 14x_2 + 5x_3 \\ S: \quad 9x_1 + 4x_2 + 4x_3 &\leq 54 \\ 9x_1 + 5x_2 + 5x_3 &\leq 63 \\ x_2 &\leq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	0	0	1	9	108
x_1	1	0	0	5/9	-4/9	0	2
x_3	0	0	1	-1	1	-1	4
x_2	0	1	0	0	0	1	5