MTM3691 - Theory of Linear Programming

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Week11

- Duality in Equation Form
- Primal Dual Relationships
- Inverse Matrix
- Optimal Dual Solution

4.1 DEFINITION OF THE DUAL PROBLEM

The **dual** problem is an LP defined directly and systematically from the **primal** (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other.

In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), types of constraints ($\leq$, $\geq$, or $=$), and orientation of the variables (nonnegative or unrestricted). This type of treatment is somewhat confusing, and for this reason we offer a *single* definition that automatically subsumes *all* forms of the primal.

Our definition of the dual problem requires expressing the primal problem in the *equation form* presented in Section 3.1 (all the constraints are equations with nonnegative right-hand side and all the variables are nonnegative).

To show how the dual problem is constructed, define the primal in *equation form* as follows:

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1, 2, \ldots, m$$

$$x_j \geq 0,\ j = 1, 2, \ldots, n$$

The variables x_j, $j = 1, 2, \ldots, n$, include the surplus, slack, and artificial variables, if any.

Table 4.1 shows how the dual problem is constructed from the primal. Effectively, we have

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficients of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

TABLE 4.1 Construction of the Dual from the Primal

	Primal variables						
	x_1	x_2	...	x_j	...	x_n	
Dual variables	c_1	c_2	...	c_j	...	c_n	Right-hand side
y_1	a_{11}	a_{12}	...	a_{1j}	...	a_{1n}	b_1
y_2	a_{21}	a_{22}	...	a_{2j}	...	a_{2n}	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	...	a_{mj}	...	a_{mn}	b_m
				↑ *j*th dual constraint			↑ Dual objective coefficients

The rules for determining the sense of optimization (maximization or minimization), the type of the constraint ($\leq$, $\geq$, or $=$), and the sign of the dual variables are summarized in Table 4.2. Note that the sense of optimization in the dual is always opposite to that of the primal. An easy way to remember the constraint type in the dual (i.e., $\leq$ or $\geq$) is that if the dual objective is *minimization* (i.e., pointing *down*), then the constraints are all of the type $\geq$ (i.e., pointing *up*). The opposite is true when the dual objective is maximization.

TABLE 4.2 Rules for Constructing the Dual Problem

Primal problem objective[a]	Dual problem		
	Objective	*Constraints type*	*Variables sign*
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

[a]All primal constraints are equations with nonnegative right-hand side, and all the variables are nonnegative.

Example 4.1-1

Primal
Maximize $z = 5x_1 + 12x_2 + 4x_3$ subject to $x_1 + 2x_2 + x_3 \leq 10$ $2x_1 - x_2 + 3x_3 = 8$ $x_1, x_2, x_3 \geq 0$

Example 4.1-2

Primal
Minimize $z = 15x_1 + 12x_2$
subject to
$x_1 + 2x_2 \geq 3$
$2x_1 - 4x_2 \leq 5$
$x_1, x_2 \geq 0$

Example 4.1-3

Primal
Maximize $z = 5x_1 + 6x_2$ subject to $x_1 + 2x_2 = 5$ $-x_1 + 5x_2 \geq 3$ $4x_1 + 7x_2 \leq 8$ x_1 unrestricted, $x_2 \geq 0$

Dual Problem Minimize $z = 5y_1 + 3y_2 + 8y_3$

subject to

$$
\begin{aligned}
y_1 - y_2 + 4y_3 &\geq 5 \\
-y_1 + y_2 - 4y_3 &\geq -5 \\
2y_1 + 5y_2 + 7y_3 &\geq 6 \\
-y_2 \qquad\quad &\geq 0 \\
y_3 &\geq 0
\end{aligned}
$$

y_1, y_2, y_1 unrestricted

The first and second constraints are replaced by an equation. The general rule in this case is that an unrestricted primal variable always corresponds to an equality dual constraint. Conversely, a primal equation produces an unrestricted dual variable, as the first primal constraint demonstrates.

Summary of the Rules for Constructing the Dual. The general conclusion from the preceding examples is that the variables and constraints in the primal and dual problems are defined by the rules in Table 4.3. It is a good exercise to verify that these explicit rules are subsumed by the general rules in Table 4.2.

TABLE 4.3 Rules for Constructing the Dual Problem

Maximization problem		Minimization problem
Constraints		*Variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		*Constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

Note that the table does not use the designation primal and dual. What matters here is the sense of optimization. If the primal is maximization, then the dual is minimization, and vice versa.

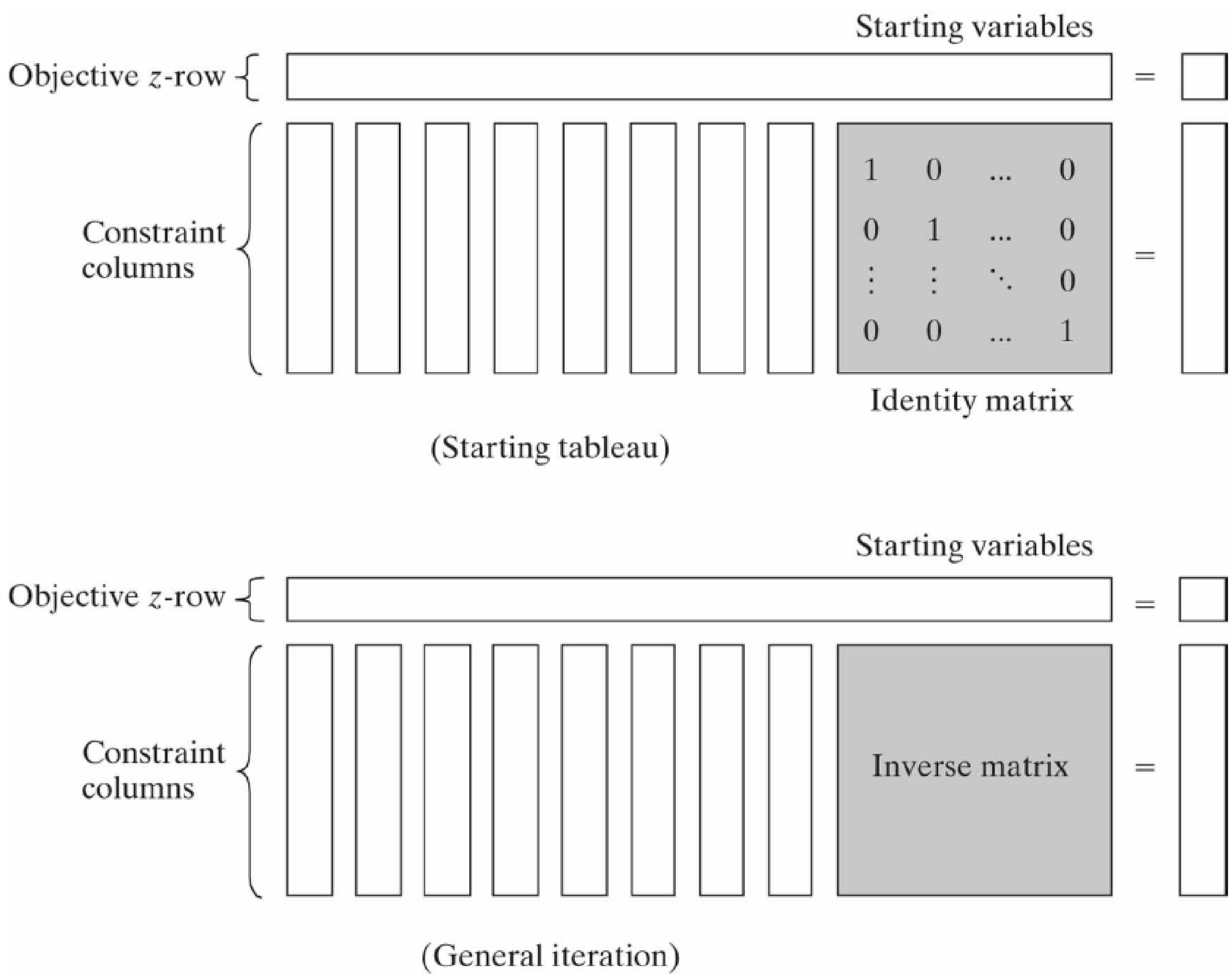
4.2 PRIMAL-DUAL RELATIONSHIPS

Changes made in the original LP model will change the elements of the current optimal tableau, which in turn may affect the optimality and/or the feasibility of the current solution. This section introduces a number of primal-dual relationships that can be used to recompute the elements of the optimal simplex tableau. These relationships will form the basis for the economic interpretation of the LP model as well as for post-optimality analysis.

4.2.2 Simplex Tableau Layout

Figure 4.1 gives a schematic representation of the *starting* and *general* simplex tableaus. In the starting tableau, the constraint coefficients under the starting variables form an **identity matrix** (all main-diagonal elements equal 1 and all off-diagonal elements equal zero). With this arrangement, subsequent iterations of the simplex tableau generated by the Gauss-Jordan row operations (see Chapter 3) will modify the elements of the identity matrix to produce what is known as the **inverse matrix**. As we will see in the remainder of this chapter, the inverse matrix is key to computing all the elements of the associated simplex tableau.

Figure 4.1 Schematic representation of the starting and general simplex tableaus



4.2.3 Optimal Dual Solution

The primal and dual solutions are so closely related that the optimal solution of either problem directly yields (with little additional computation) the optimal solution to the other. Thus, in an LP model in which the number of variables is considerably smaller than the number of constraints, computational savings may be realized by solving the dual, from which the primal solution is determined automatically. This result follows because the amount of simplex computation depends largely (though not totally) on the number of constraints (see Problem 2, Set 4.2c).

This section provides two methods for determining the dual values. Note that the dual of the dual is itself the primal, which means that the dual solution can also be used to yield the optimal primal solution automatically.

Method 1.

$$\begin{pmatrix}\text{Optimal value of} \\ \textit{dual} \text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient of } \textit{starting} \text{ variable } x_i \\ + \\ \textit{Original} \text{ objective coefficient of } x_i\end{pmatrix}$$

Method 2.

$$\begin{pmatrix}\text{Optimal values} \\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of} \\ \text{original objective coefficients} \\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal} \\ \text{inverse}\end{pmatrix}$$

The elements of the row vector must appear in the same order in which the basic variables are listed in the *Basic* column of the simplex tableau.

Example 4.2-1

Consider the following LP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 10 \\ 2x_1 - x_2 + 3x_3 &= 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

To prepare the problem for solution by the simplex method, we add a slack x_4 in the first constraint and an artificial R in the second. The resulting primal and the associated dual problems are thus defined as follows:

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR$	Minimize $w = 10y_1 + 8y_2$
subject to	subject to
$x_1 + 2x_2 + x_3 + x_4 = 10$	$y_1 + 2y_2 \ge 5$
$2x_1 - x_2 + 3x_3 + R = 8$	$2y_1 - y_2 \ge 12$
$x_1, x_2, x_3, x_4, R \ge 0$	$y_1 + 3y_2 \ge 4$
	$y_1 \ge 0$
	$y_2 \ge -M$ ($\Rightarrow y_2$ unrestricted)

TABLE 4.4 Optimal Tableau of the Primal of Example 4.2-1

Basic	x_1	x_2	x_3	x_4	R	Solution
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5} + M$	$54\frac{4}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Table 4.4 provides the optimal primal tableau.

We now show how the optimal dual values are determined using the two methods described at the start of this section.

Method 1. In Table 4.4, the starting primal variables x_4 and R uniquely correspond to the dual variables y_1 and y_2, respectively. Thus, we determine the optimum dual solution as follows:

Starting primal basic variables	x_4	R
z-equation coefficients	$\frac{29}{5}$	$-\frac{2}{5} + M$
Original objective coefficient	0	$-M$
Dual variables	y_1	y_2
Optimal dual values	$\frac{29}{5} + 0 = \frac{29}{5}$	$-\frac{2}{5} + M + (-M) = -\frac{2}{5}$

Method 2. The optimal inverse matrix, highlighted under the starting variables x_4 and R, is given in Table 4.4 as

First, we note that the optimal primal variables are listed in the tableau in *row order* as x_2 and then x_1. This means that the elements of the original objective coefficients for the two variables must appear in the same order—namely,

$$\begin{aligned}(\text{Original objective coefficients}) &= (\text{Coefficient of } x_2, \text{coefficient of } x_1)\\ &= (12, 5)\end{aligned}$$

Thus, the optimal dual values are computed as

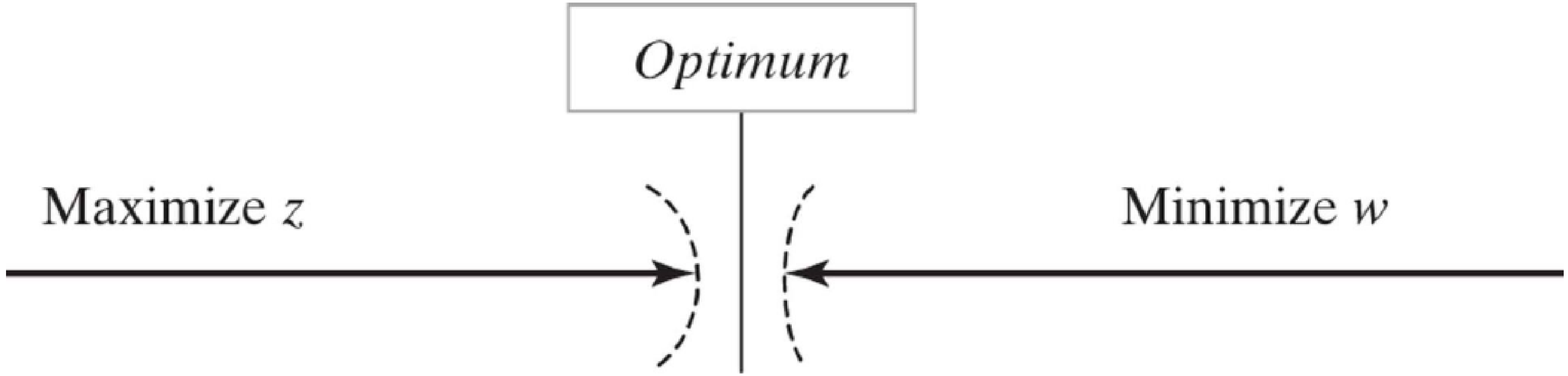
Primal-dual objective values. Having shown how the optimal dual values are determined, next we present the relationship between the primal and dual objective values. For any pair of *feasible* primal and dual solutions,

$$\begin{pmatrix}\text{Objective value in the} \\ \textit{maximization} \text{ problem}\end{pmatrix} \leq \begin{pmatrix}\text{Objective value in the} \\ \textit{minimization} \text{ problem}\end{pmatrix}$$

At the optimum, the relationship holds as a strict equation. The relationship does not specify which problem is primal and which is dual. Only the sense of optimization (maximization or minimization) is important in this case.

The optimum cannot occur with z strictly less than w (i.e., $z < w$) because, no matter how close z and w are, there is always room for improvement, which contradicts optimality as Figure 4.2 demonstrates.

Figure 4.2 Relationship between maximum *z* and minimum *w*



Example 4.2-2

In Example 4.2-1, $\left(x_1 = 0, x_2 = 0, x_3 = \frac{8}{3}\right)$ and $(y_1 = 6, y_2 = 0)$ are feasible primal and dual solutions. The associated values of the objective functions are

$$z = 5x_1 + 12x_2 + 4x_3 = 5(0) + 12(0) + 4\left(\frac{8}{3}\right) = 10\tfrac{2}{3}$$

$$w = 10y_1 + 8y_2 = 10(6) + 8(0) = 60$$

Thus, $z\left(= 10\frac{2}{3}\right)$ for the maximization problem (primal) is less than w $(= 60)$ for the minimization problem (dual). The optimum value of $z\left(= 54\frac{4}{5}\right)$ falls within the range $\left(10\frac{2}{3}, 60\right)$.