MTM3691 - Theory of Linear Programming

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Week13

- Dual Feasibility Condition
- Dual Optimality Condition
- Dual Simplex Algorithm
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4.4 ADDITIONAL SIMPLEX ALGORITHMS

In the simplex algorithm presented in Chapter 3 the problem starts at a (basic) feasible solution. Successive iterations continue to be feasible until the optimal is reached at the last iteration. The algorithm is sometimes referred to as the **primal simplex** method.

This section presents two additional algorithms: The **dual simplex** and the **generalized simplex**. In the dual simplex, the LP starts at a better than optimal *infeasible* (basic) solution. Successive iterations remain infeasible and (better than) optimal until feasibility is restored at the last iteration. The generalized simplex combines both the primal and dual simplex methods in one algorithm. It deals with problems that start both nonoptimal and infeasible. In this algorithm, successive iterations are associated with basic feasible or infeasible (basic) solutions. At the final iteration, the solution becomes optimal and feasible (assuming that one exists).

All three algorithms, the primal, the dual, and the generalized, are used in the course of post-optimal analysis calculations, as will be shown in Section 4.5.

4.4.1 Dual Simplex Algorithm

The crux of the dual simplex method is to start with a better than optimal and infeasible basic solution. The optimality and feasibility conditions are designed to preserve the optimality of the basic solutions while moving the solution iterations toward feasibility.

Dual feasibility condition. The leaving variable, x_r, is the basic variable having the most negative value (ties are broken arbitrarily). If all the basic variables are nonnegative, the algorithm ends.

Dual optimality condition. Given that x_r is the leaving variable, let $\bar{c}_j$ be the reduced cost of nonbasic variable x_j and α_{rj} the constraint coefficient in the x_r-row and x_j-column of the tableau. The entering variable is the nonbasic variable with $\alpha_{rj} < 0$ that corresponds to

$$\min_{\text{Nonbasic } x_j} \left\{ \left|\frac{\bar{c}_j}{\alpha_{rj}}\right|, \alpha_{rj} < 0 \right\}$$

(Ties are broken arbitrarily.) If $\alpha_{rj} \geq 0$ for all nonbasic x_j, the problem has no feasible solution.

To start the LP optimal and infeasible, two requirements must be met:

1. The objective function must satisfy the optimality condition of the regular simplex method (Chapter 3).
2. All the constraints must be of the type ($\leq$).

The second condition requires converting any ($\geq$) to ($\leq$) simply by multiplying both sides of the inequality ($\geq$) by -1. If the LP includes ($=$) constraints, the equation can be replaced by two inequalities. For example,

$$x_1 + x_2 = 1$$

is equivalent to

$$x_1 + x_2 \leq 1, x_1 + x_2 \geq 1$$

or

$$x_1 + x_2 \leq 1, -x_1 - x_2 \leq -1$$

After converting all the constraints to ($\leq$), the starting solution is infeasible if at least one of the right-hand sides of the inequalities is strictly negative.

Example 4.4-1 Minimize $z = 3x_1 + 2x_2 + x_3$

subject to

$$\begin{aligned} 3x_1 + x_2 + x_3 &\geq 3 \\ -3x_1 + 3x_2 + x_3 &\geq 6 \\ x_1 + x_2 + x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In the present example, the first two inequalities are multiplied by -1 to convert them to ($\leq$) constraints. The starting tableau is thus given as:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6
x_6	1	1	1	0	0	1	3

The tableau is optimal because all the reduced costs in the z-row are ≤ 0 ($\bar{c}_1 = -3, \bar{c}_2 = -2, \bar{c}_3 = -1, \bar{c}_4 = 0, \bar{c}_5 = 0, \bar{c}_6 = 0$). It is also infeasible because at least one of the basic variables is negative ($x_4 = -3, x_5 = -6, x_6 = 3$).

According to the dual feasibility condition, x_5 ($= -6$) is the leaving variable. The next table shows how the dual optimality condition is used to determine the entering variable.

	$j = 1$	$j = 2$	$j = 3$
Nonbasic variable	x_1	x_2	x_3
z-row ($\bar{c}_j$)	-3	-2	-1
x_5-row, α_{4j}	3	-3	-1
Ratio, $\left\|\frac{\bar{c}_j}{\alpha_{5j}}\right\|$, $\alpha_{5j} < 0$	—	$\frac{2}{3}$	1

The ratios show that x_2 is the entering variable. Notice that a nonbasic variable x_j is a candidate for entering the basic solution only if its α_{rj} is strictly negative. This is the reason x_1 is excluded in the table above.

The next tableau is obtained by using the familiar row operations, which give

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
x_4	-4	0	$-\frac{2}{3}$	1	$-\frac{1}{3}$	0	-1
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
Ratio	$\frac{5}{4}$	—	$\frac{1}{2}$	—	2	—	

The preceding tableau shows that x_4 leaves and x_3 enters, thus yielding the following tableau, which is both optimal and feasible:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{9}{2}$
x_3	6	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
x_2	-3	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_6	-2	0	0	1	0	1	0

Notice how the dual simplex works. In all the iterations, optimality is maintained (all reduced costs are ≤ 0). At the same time, each new iteration moves the solution toward feasibility. At iteration 3, feasibility is restored for the first time and the process ends with the optimal feasible solution given as $x_1 = 0$, $x_2 = \frac{3}{2}$, $x_2 = \frac{3}{2}$, and $z = \frac{9}{2}$.

4.4.2 Generalized Simplex Algorithm

The (primal) simplex algorithm in Chapter 3 starts feasible but nonoptimal. The dual simplex in Section 4.4.1 starts (better than) optimal but infeasible. What if an LP model starts both nonoptimal and infeasible? We have seen that the primal simplex accounts for the infeasibility of the starting solution by using artificial variables. Similarly, the dual simplex accounts for the nonoptimality by using an artificial constraint (see Problem 3, Set 4.4a). Although these procedures are designed to enhance *automatic* computations, such details may cause one to lose sight of what the simplex algorithm truly entails—namely, the optimum solution of an LP is associated with a corner point (or basic) solution. Based on this observation, you should be able to "tailor" your own simplex algorithm for LP models that start both nonoptimal and infeasible. The following example illustrates what we call the generalized simplex algorithm.

Example 4.4-2

Consider the LP model of Problem 4(a), Set 4.4a. The model can be put in the following tableau form in which the starting basic solution (x_3, x_4, x_5) is both nonoptimal (because x_3 has a negative reduced cost) and infeasible (because $x_4 = -8$). (The first equation has been multiplied by -1 to reveal the infeasibility directly in the *Solution* column.)

Maximize $z = 2x_3$
subject to

$$-x_1 + 2x_2 - 2x_3 \geq 8$$
$$-x_1 + x_2 + x_3 \leq 4$$
$$2x_1 - x_2 + 4x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	−2	0	0	0	0
x_4	1	−2	2	1	0	0	−8
x_5	−1	1	1	0	1	0	4
x_6	2	−1	4	0	0	1	10

We can solve the problem without the use of any artificial variables or artificial constraints as follows: Remove infeasibility first by applying a version of the dual simplex feasibility condition that selects x_4 as the leaving variable. To determine the entering variable, all we need is a nonbasic variable whose constraint coefficient in the x_4-row is strictly negative. The selection can be done without regard to optimality, because it is nonexistent at this point anyway (compare with the dual optimality condition). In the present example, x_2 has a negative coefficient in the x_4-row and is selected as the entering variable. The result is the following tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	-2	0	0	0	0
x_2	$-\frac{1}{2}$	1	-1	$-\frac{1}{2}$	0	0	4
x_5	$-\frac{1}{2}$	0	2	$\frac{1}{2}$	1	0	0
x_6	$\frac{3}{2}$	0	3	$-\frac{1}{2}$	0	1	14

The solution in the preceding tableau is now feasible but nonoptimal, and we can use the primal simplex to determine the optimal solution. In general, had we not restored feasibility in the preceding tableau, we would repeat the procedure as necessary until feasibility is satisfied or there is evidence that the problem has no feasible solution (which happens if a basic variable is negative and all its constraint coefficients are nonnegative). Once feasibility is established, the next step is to pay attention to optimality by applying the proper optimality condition of the primal simplex method.

Remarks. The essence of Example 4.4-2 is that the simplex method is not rigid. The literature abounds with variations of the simplex method (e.g., the primal-dual method, the symmetrical method, the criss-cross method, and the multiplex method) that give the impression that each procedure is different, when, in effect, they all seek a corner point solution, with a slant toward automated computations and, perhaps, computational efficiency.