MTM3691-Theory of Linear Programming

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Week 2

Introduction to Linear Programming: Properties of LP Model

In this example, the objective functions and the constraints are all linear. The linearity assumption is equivalent to the following two properties:

1) **(Proportionality):** This feature states that the effect of each decision variable on both the objective function and all constraints should be directly proportional to the value of the corresponding variable. For example in Reddy Mikks problem, if there is a discount when the sales exceed certain values, the revenues will not be proportional to the increase in the sales amount.
2) **(Additivity):** This feature stipulates that the total contribution of all variables in the constraints and the objective function consists of the sum of the individual contributions of these variables. For example, if there are two *competing products*, this feature will not be realized, as increasing the sales of one of them will adversely affect the sales of the other.

The graphical procedure includes two steps:

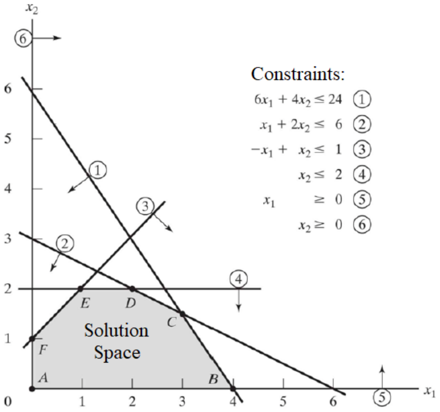
1) Determination of the feasible solution space,
2) Determining the optimum solution among all the feasible points in the feasible solution space.

The method is explained for cases where the objective function is both maximum and minimum.

Example

Let us consider the Reddy Mikks problem again.

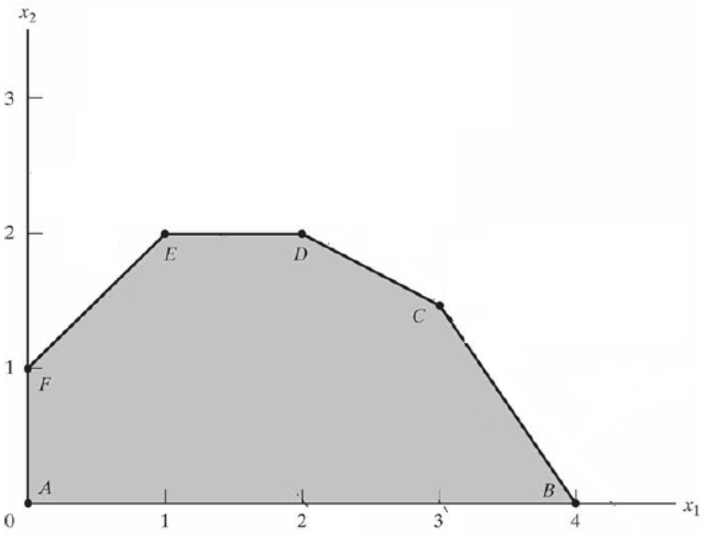
- **Step 1 (Determination of the Feasible Solution Space):**



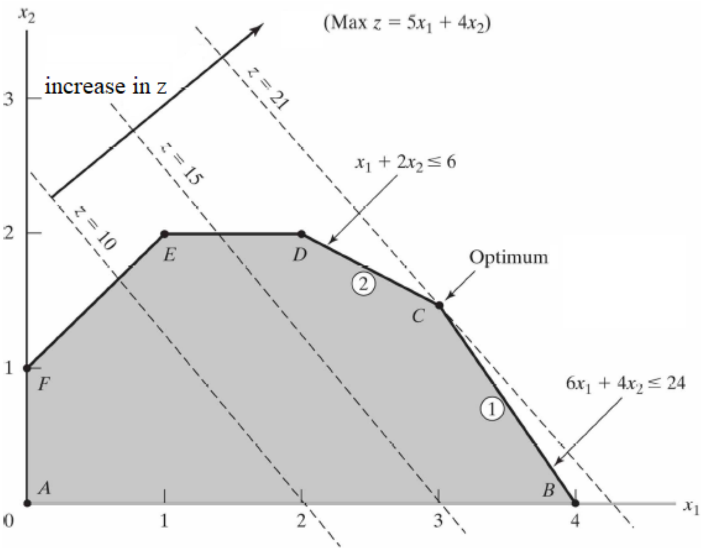
- **Step 2 (Determination of the Optimum Solution):** The solution space is formed by line segments connecting the vertices *A*, *B*, *C*, *D*, *E* and *F*. Any point on or within the boundaries of *ABCDEF* space is a feasible solution that satisfies all constraints. Since *ABCDEF* feasible space contains an infinite number of suitable points, a systematic method is needed to find the optimum point.

  In order to determine the optimum solution, the direction of increase of the objective function in the form $z = 5x_1 + 4x_2$ (since the goal is to be maximized) must be known. This can be done in practice by arbitrarily increasing the values of z. For example, for the levels $z = 10$, $z = 15$ and $z = 21$, the lines $5x_1 + 4x_2 = 10$, $5x_1 + 4x_2 = 15$ and $5x_1 + 4x_2 = 21$ are the equivalent level lines. The inclusion of this to the solution of the model and the direction of increase of z can be determined as shown in the following figure.

Introduction to Linear Programming: Graphical Solution (Max)



Introduction to Linear Programming: Graphical Solution (Max)

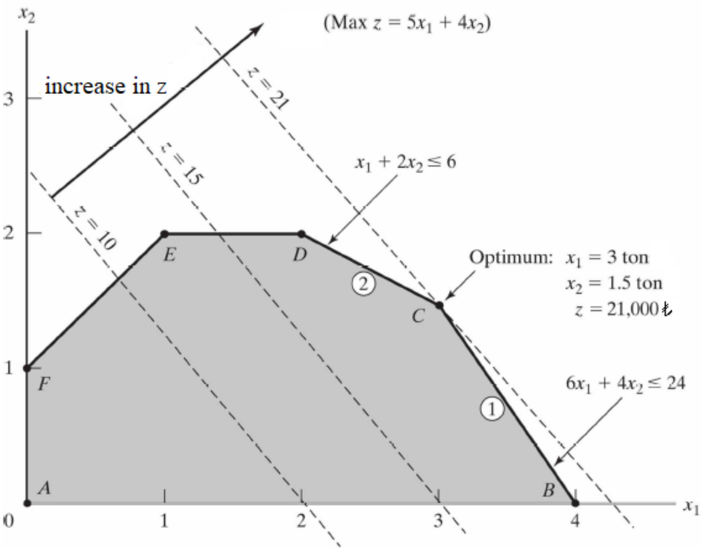


- **Step 2 (Determining the Optimum Solution-Continued):** It can be seen from the figure that the solution space exits at point C. Therefore, the optimum solution will occur at .

  The point C is the intersection point of the lines shown with numbers 1 and 2 in the figure, i.e.

  The solution of these equations of these lines will give $x_1 = 3$ and $x_2 = 1.5$. The objective function is calculated as $z = 5 \times 3 + 4 \times 1.5 = 21$. So, if 3 tons of exterior paint and 1.5 tons of interior paint are produced in daily production, the associated daily profit will be 21000 TL.

Introduction to Linear Programming: Graphical Solution (Max)



Introduction to Linear Programming: Graphical Solution (Max)

It is not a coincidence that the optimum solution is at a corner point of the solution space. This is valid even when the objective function z is parallel to any constraint. For example, if the objective function was $z = 6x_1 + 4x_2$ (parallel to the constraint expressed by 1), corner points B or C would be found as the optimum solution. In fact, any point of the line segment BC would be an *alternative optimum* (see: (Taha, 2010, Example 3.5.2)).

This observation indicates that the LP optimum is always at the corner points and therefore the optimum solution can be found by examining the objective function values at the corner points of the region obtained from the constraints.

Corner Point	(x_1, x_2)	z
A	$(0, 0)$	
B	$(4, 0)$	
C	$(3, 1.5)$	
D	$(2, 2)$	
E	$(1, 2)$	
F	$(0, 1)$	

- **Method 1: Enumerating all the corner points**
  - Determine the feasible space.
  - List all the corner points.
  - By evaluating the z values at the corner points, select the 'best' as optimum.
- **Method 2: Determining the increasing/decreasing directions of z**
  - By assigning two arbitrary values to z, plot the iso-profit/iso-cost lines corresponding to these values.
  - Determine the direction.
  - Lines are drawn in the specified direction, parallel to the drawn z lines.
  - The last z line that intersects with the solution space is determined and this z line indicates the best z value, that is, the optimal solution of the LP.

- **Method 3: Gradient method:** As we know from analysis courses, moving in the direction of the gradient ($\vec{c}$) increases the z values. Similarly, when moving in the opposite direction of the gradient ($-\vec{c}$) objective function values decrease.

  That is why, in maximization problems, iso-profit lines are drawn in the direction of the gradient ($\vec{c}$) whereas in minimization problems, iso-cost lines are drawn in the opposite direction of the gradient ($-\vec{c}$). Finally, the last point obtained from the intersection of z lines with the solution space is determined as the optimum solution.

Example (Diet Problem)

At least 800 kg of silage (a special type of feed) is used per day in Ozark Farms. This special type of feed is obtained from a mixture of corn and soybean meal with the following compositions:

	kg per 1 kg of feedstuff		Cost (TL/kg)
	Protein	Fiber	
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

The composition of this special feed must contain at least 30 % protein and at most 5 % fiber. Ozark company wants to determine the minimum cost of the daily feed mix.

Since the feed consists of corn and soybean meal, the decision variables in the model are defined as follows:

Introduction to Linear Programming: Graphical Solution (Min)

The objective function here seeks the minimum of the total daily feed cost (in TL):

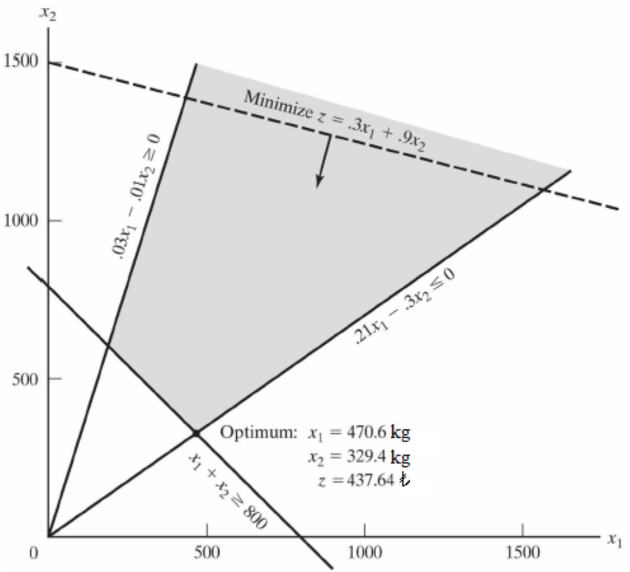
The constraints of the model should reflect both the amount of daily requirement and the characteristics of the mixture. Considering that Ozark Farms needs at least 800 kg of feed per day, we can express this constraint as follows:

Restrictions on protein diet requirements are also given below: The amount of protein in x_1 kg corn and x_2 kg soybean meal is $0.09x_1 + 0.6x_2$ kg. Since this amount is desired to be at least 30 % of the total product mix, we have

Similarly, the amount of fiber will be

When we rearrange all the constraints, the model with the objective function can be expressed as follows:

Introduction to Linear Programming: Graphical Solution (Min)



Course Content

- Chapter 2: Introduction to Linear Programming (LP)
- Chapter 3: The Simplex Method
  - LP Model in Equation Form
  - Transition from Graphical Solution to Algebraic Solution
  - Algebraic Method
- Chapter 4: Duality and Sensitivity Analysis
- Chapter 5: Transportation Model and Various Transportation Models

The Simplex Method: LP Model in Equation Form

The problem must be standardized before basic solutions can be used when solving the general LP model. For a model to become standard, it must provide the following features:

1) All constraints, except those that require the variables to be non-negative, must be transformed into equations with a non-negative right hand side.
2) All variables must be zero or positive (not negative).
3) The objective function can be maximum or minimum.

Warning: Commercial softwares that uses the simplex method (e.g. TORA, LINDO, etc.) generally accepts inputs conforming to this standard form. In this context, the operating instructions should be read carefully before using these programs.

The Simplex Method: LP Model in Equation Form

1) **Turning inequalities into equalities:** An inequality of the form $\leq$ (or $\geq$) can be converted to an equality by adding a slack variable to the left hand side:
   - **For a constraint of the form $\leq$:**

     $$x_1 + 2x_2 \leq 3 \Leftrightarrow$$

     Here $s_1 \geq 0$ and it is called a **slack variable**.
   - **For a constraint of the form $\geq$:**

     $$3x_1 + x_2 \geq 5 \Leftrightarrow$$

     Here $S_1 \geq 0$ and it is called a **surplus variable**.

   The right-hand side of an equation must satisfy the non-negative condition. In fact, if necessary, the equation can be multiplied by -1 to achieve this. Another point that should be noted here is that an inequality of the form ($\leq$) can be transformed into an inequality of the form ($\geq$) by multiplying both sides by -1.

The Simplex Method: LP Model in Equation Form

2) **Converting unbounded variables into non-negative variables:** An unbounded variable x_j can be expressed in terms of two non-negative variables as shown below:

$$x_j = x_j^+ - x_j^-, \quad x_j^+, x_j^- \geq 0.$$

Warning: The process of substituting one variable for another as shown above will also affect all constraints and objective function. Once the problem is solved with variables x_j^+ and x_j^-, the value of the original variable must be determined by the operation $x_j = x_j^+ - x_j^-$.

3) **Converting maximization to minimization:** Maximization of a function $f(x_1, x_2, ..., x_n)$ is equivalent to minimization of the function $-f(x_1, x_2, ..., x_n)$. Both problems have the same optimal values of x_1, x_2, ..., x_n.

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

Graphical Method	Algebraic Method
Graph all constraints, including nonnegativity restrictions	Represent the solution space by m equations in n variables and restrict all variables to nonnegative values, $m < n$
Solution space consists of infinity of **feasible** points	The system has infinity of **feasible** solutions
Identify **feasible corner points** of the solution space	Determine the **feasible basic solutions** of the equations
Candidates for the optimum solution are given by a *finite* number of **corner points**	Candidates for the optimum solution are given by a *finite* number of **basic feasible solutions**
Use the objective function to determine the **optimum corner point** from among all the candidates	Use the objective function to determine the **optimum basic feasible solution** from among all the candidates

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

In the graphical method, since the solution space is divided into half-spaces determined by inequality constraints, there are an infinite number of solution points. In the algebraic method, since the solution space is expressed by m simultaneous linear equations containing n variables, if $m < n$, there is still an infinite number of solution points.

In this case ($m < n$); $n - m$ variables will take the value of zero, and the value of m variables will be determined by solving m equations.

If these m variables give *a unique solution*, the variables that satisfy this situation are called **basic variables**, and $n - m$ zero-valued variables are called **non-basic variables**.

In this case, the situation that results in a singular solution also includes **basic solution**. If we assume that all variables take non-negative values, then the basic solution is **feasible**. Otherwise, it is **infeasible**.

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

Based on the given definitions, the maximum number of basic solutions (or vertices) for m equations with n unknowns will be evaluated as follows?

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

Let us explain this with the following example.

Example

Let us consider the following two variable LP:

$$\begin{aligned} \max z &= 2x_1 + 3x_2 \\ \textit{subject to } 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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Algebraically, the solution space of this LP is as follows.

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

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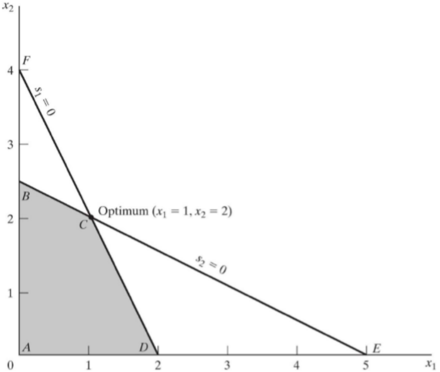

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

Now, for the system below, let us calculate the combinations of the basic feasible solution, the basic infeasible solution and the non-basic solution variables.

Number of all possible basic solutions (vertices)

$$\binom{4}{2} = \frac{4!}{2!2!} = 6.$$

Here,

- $n - m = 4 - 2 = 2$ is the number of variables that will take the value zero,
- $m = 2$ indicates the number of variables whose value will be determined by solving the equation.

The Simplex Method:

Transition from Graphical Solution to Algebraic Solution

Non-Basic Variables	Basic Variables	Basic Solution	Corner Point	Feasible?	Objective Function Value
(x_1, x_2)					
(x_1, s_1)					
(x_1, s_2)					
(x_2, s_1)	(x_1, s_2)	$(2, 3)$	D	Yes	
(x_2, s_2)	(x_1, s_1)	$(5, -6)$	E	No	
(s_1, s_2)	(x_1, x_2)	$(1, 2)$	C	Yes	

The Simplex Method (Summary)

- **Step 1:** By adding slack/surplus variables, inequalities are converted to equality form.
- **Step 2:** For m equations and n variables; by determining $n - m$ non-basic (zero) variables, a solution is found for the remaining m basic variables. If the basic solution is feasible, the objective function value is calculated.
- **Step 3:** For all basic variable combinations **Step 2** is applied.
- **Step 4:** The optimum solution is determined according to the maximum/minimum values of the objective function.