MTM3691-Theory of Linear Programming

Gökhan Göksu, PhD

Week 5

Course Content

- Chapter 2: Introduction to Linear Programming (LP)
- Chapter 3: Simplex Method
  - Penalty Rule for Artificial Vaiables
  - M-Method
- Chapter 4: Duality and Sensitivity Analysis
- Chapter 5: Transportation Model and Various Transportation Models

Artificial Starting Solution

Simplex iterations that start with a feasible basic solution guarantee that subsequent iterations will also be feasible. As in the examples shown so far, slack variables provide a starting feasible basic solution for LP problems where all constraints are of type $\leq$ with non-negative right-hand side constants. This situation is not met in models with constraints such as $=$ and $\geq$.

LP problems with $\leq$ conditions were solved by transforming them into equality form with the help of slack variables. Constraints of the form $=$ and $\geq$ cannot be solved with the help of slack variables. The most common procedure for generating an starting feasible basic solution for LP problems without feasible slack variables is to use **artificial variables**. These variables are assumed to play the role of slack variables in the first iteration. In this context, we will examine the method known as the *M*-method in the literature.

Artificial Starting Solution: M-Method

In the M-method, the solution starts with the standard form. For any equality i that does not have a slack variable, an artificial variable R_i is added. Since artificial variables are actually the variables that enter the model later, these variables have a very big **penalty** coefficient in the objective function to ensure that they go out of the set of basic variables by taking the value of zero in the next iterations of the simplex algorithm.

Suppose that this penalty coefficient M is a sufficiently large, positive number (mathematically $M \to \infty$). If the objective function is maximum, the coefficient of the artificial variable R_i will be $-M$ (i.e. $-MR_i$), and if it is minimum, it will be $+M$ (i.e. $+MR_i$):

$$\begin{matrix}\text{Artificial variable} \\ \text{objective coefficient}\end{matrix} = \begin{cases} -M, & \text{in maximization problems,} \\ M, & \text{in minimization problems.} \end{cases}$$

Due to the nature of optimization problems, R_i will try to get the value zero during simplex iterations due to this penalty.

Artificial Starting Solution: M-Method

Example

$$
\begin{aligned}
\min z = 4x_1 + \ \ x_2 & \\
\textit{subject to}\ 3x_1 + \ \ x_2 &= 3 \\
4x_1 + 3x_2 &\geq 6 \\
x_1 + 2x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

In order to convert the problem in standard form, the surplus variable x_3 will be removed from the second constraint, while the slack variable x_4 will be added to the third constraint. We obtain the following.

Artificial Starting Solution: M-Method

Since the variables in the first and second equations do not have slack variables, artificial variables R_1 and R_2 will be added in these two equations, and there will be a $MR_1 + MR_2$ penalty in the objective function (because we are minimizing). As a result, the LP problem becomes as the following.

The starting basic solution in this problem is $(R_1, R_2, x_4) = (3, 6, 4)$. Here, theoretically the implementation of the M-method requires $M \to \infty$. However, for computational concerns, M must be *finite* and *large enough*. This "large enough" statement is an open-ended question. What is certain is that M functions as a large enough penalty. Because if M is too large, it will also impair the accuracy of the calculations. In such cases, the main concern should be on rounding errors that occur when large and small numbers are mixed together in the model. For example, in this example, choosing $M = 100$ may be a reasonable choice.

Artificial Starting Solution: M-Method

Before creating the initial simplex tableau, the artificial variables R_1 and R_2 need to be withdrawn from the constraints and replaced in the objective function. We obtain the following.

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \quad \Longrightarrow \quad R_1 = 3 - 3x_1 - x_2, \\ 4x_1 + 3x_2 - x_3 + R_2 &= 6 \quad \Longrightarrow \quad R_2 = 6 - 4x_1 - 3x_2 + x_3. \end{aligned}$$

If we replace these values of the artificial variables in the objective function z, the following is obtained.

$$\begin{aligned} z &= 4x_1 + x_2 + M(3 - 3x_1 - x_2) + M(6 - 4x_1 - 3x_2 + x_3) \\ &= x_1(4 - 7M) + x_2(1 - 4M) + Mx_3 + 9M \\ \Longrightarrow \quad & z - (4 - 7M)x_1 - (1 - 4M) - Mx_3 = 9M \end{aligned}$$

Therefore, starting simplex tableau[1] will be as follows:

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	-4+7M	-1+4M	-M	0	0	0	9M
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

[1] **Note:** Since the z column remains the same across all iterations, it has been removed for simplicity.

Artificial Starting Solution: M-Method

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	-4+7M	-1+4M	-M	0	0	0	9M
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Since the objective function is to be minimized, the variable with the largest positive coefficient in the z row, i.e. x_1, will be the entering variable. Next, the ratio tableau is to be checked for the feasibility condition:

Basic	x_1	Solution	Ratio
R_1	3	3	
R_2	4	6	$x_1 = 6/4 = 1.5$
s_4	1	4	$x_1 = 4/1 = 4$

According to the feasibility condition, R_1 is the leaving variable. The following new simplex tableau is obtained after Gauss-Jordan row operations:

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	(1+5M)/3	-M	(4-7M)/3	0	0	4+2M
x_1	1	1/3	0	1/3	0	0	1
R_2	0	5/3	-1	-4/3	1	0	2
x_4	0	5/3	0	-1/3	0	1	3

Artificial Starting Solution: M-Method

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	(1+5M)/3	-M	(4-7M)/3	0	0	4+2M
x_1	1	1/3	0	1/3	0	0	1
R_2	0	5/3	-1	-4/3	1	0	2
x_4	0	5/3	0	-1/3	0	1	3

Since x_2 is the only positive coefficient variable in the z row, x_2 will be the entering variable. For the feasibility condition, the ratio tableau is again to be checked.

Basic	x_2	Solution	Ratio
x_1	1/3	1	$x_2 = 1/(1/3) = 3$
R_2	5/3	2	
x_4	5/3	3	$x_2 = 3/(5/3) = 9/5$

According to the feasibility condition, R_2 will be the leaving variable.

Artificial Starting Solution: M-Method

The following simplex tableau is obtained after Gauss-Jordan row operations:

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	0	1/5	8/5-M	-1/5-M	0	18/5
x_1	1	0	1/5	3/5	-1/5	0	3/5
x_2	0	1	-3/5	-4/5	3/5	0	6/5
x_4	0	0	1	1	-1	1	1

WARNING!!! There is still a variable with a positive coefficient in the z-row. Therefore, the positive coefficient variable x_3 in the z-row will be the entering variable. For the feasibility condition, the ratio tableau is again to be checked.

Basic	x_3	Solution	Ratio
x_1	1/5	3/5	$x_3 = (3/5)/(1/5) = 3$
x_2	-3/5	6/5	$x_3 = (6/5)/(-3/5) = -2$ (ignore)
x_4	1	1	

According to the feasibility condition, x_4 will be the leaving variable.

Artificial Starting Solution: M-Method

The following simplex tableau is obtained after Gauss-Jordan row operations:

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	0	0	7/5-M	-M	-1/5	17/5
x_1	1	0	0	2/5	0	-1/5	2/5
x_2	0	1	0	-1/5	0	3/5	9/5
x_3	0	0	1	1	-1	1	1

None of the coefficients in the z-row are positive anymore. According to the optimality condition, the table is optimal. The optimal solution is $x_1 = \frac{2}{5}$, $x_2 = \frac{9}{5}$ and $x_3 = 1$, and the corresponding objective function value is $z = \frac{17.}{5}$.

WARNING!!! If the LP problem does not have a feasible solution (due to constraints not being consistent, etc.); the use of the penalty coefficient M cannot force at least one artificial variable to take the value zero in the last simplex iteration. Therefore, in such cases the resulting simplex iteration will contain at least one positive artificial variable. This indicates that there is no feasible solution to the problem. On the other hand, if all artificial variables are zero, the solution is optimal.

Artificial Starting Solution: M-Method (Examples)

Example

$$\begin{aligned} \max z &= 5x_1 + 4x_2 \\ \textit{subject to } x_1 + 2x_2 &\geq 6 \\ 4x_1 + 2x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Artificial Starting Solution: M-Method (Examples)

Example

$$\begin{aligned} \min z = -3x_1 + 4x_2 & \\ \textit{subject to } x_1 + x_2 &\leq 4 \\ 2x_1 + 3x_2 &\geq 18 \\ x_1, x_2 &\geq 0 \end{aligned}$$