MTM3691-Theory of Linear Programming

Gökhan Göksu, PhD

Week 7

Course Content

- Chapter 2: Introduction to Linear Programming (LP)
- Chapter 3: The Simplex Method
  - Standard Form
    - Free Variables with Upper Bounds
    - Free Variables with Lower Bounds
    - Variables with Interval Bounds
    - Free Variables
- Chapter 4: Duality and Sensitivity Analysis
- Chapter 5: Transportation Model and Various Transportation Models

Standard Form

LP problems need to be converted to standard form. In a standard form,

- all variables must satisfy the non-negativity condition,
- all constraints must be in equality form and have a non-negative, constant right-hand side.

When converting to standard form, it may be necessary to define a new variable or rearrange the constraints:

- Inequalities are multiplied by -1 to get non-negative right-hand side.
- Inequalities are converted into equations by adding or subtracting non-negative variables (slack or surplus).
- Unrestricted variables are rewritten with two new non-negative variables.

All LP problems can be written in standard form!!!

Standard Form: Free Variables with Upper Bounds

Free variables with upper bounds of the form $x_i \leq u_i$ are converted to standard form as follows:

$$x_i \leq u_i \quad \Longrightarrow \quad 0 \leq u_i - x_i \quad \Longrightarrow \quad x_i' \geq 0 \text{ where } x_i' = u_i - x_i$$

Example

$$\begin{aligned} \max z &= x_1 + x_2 \\ \textit{subject to} \; -x_1 + 2x_2 &\leq 8 \\ x_1 + 2x_2 &\leq 10 \\ x_1 \leq 8, \; x_2 &\geq 0 \end{aligned}$$

Standard Form: Free Variables with Upper Bounds

Example

$$
\begin{aligned}
\max z &= -x_1 + 2x_2 \\
\textit{subject to} \; -3x_1 + 4x_2 &\leq 24 \\
x_1 + 2x_2 &\leq 16 \\
0 \leq x_1 \leq 9, \;\; x_2 &\geq 0
\end{aligned}
$$

Standard Form: Free Variables with Lower Bounds

Free variables with lower bounds of the form $\ell_i \leq x_i$ are converted to standard form as follows:

$$\ell_i \leq x_i \quad \Longrightarrow \quad 0 \leq x_i - \ell_i \quad \Longrightarrow \quad x_i' \geq 0 \text{ where } x_i' = x_i - \ell_i$$

Example

$$\begin{aligned} \max z = -x_1 + 3x_2 & \\ \textit{subject to } -2x_1 + 3x_2 &\leq 24 \\ x_1 + 2x_2 &\leq 8 \\ x_1 \geq -8,\ x_2 &\geq 0 \end{aligned}$$

Standard Form: Variables with Interval Bounds

Variables with interval bounds of the form $\ell_i \le x_i \le u_i$ are converted to standard form as follows:

$$\begin{aligned} \ell_i \le x_i \le u_i \quad \Longrightarrow \quad 0 \le x_i - \ell_i \le u_i - \ell_i \quad & \Longrightarrow \quad 0 \le x_i' \le u_i - \ell_i \\ & \Longrightarrow \quad x_i' \ge 0,\ \ x_i'' \ge 0, \\ & \qquad\ \ \text{where } x_i'' = u_i - \ell_i - x_i' \end{aligned}$$

Example

$$\begin{aligned} \max z &= -x_1 + 3x_2 \\ \textit{subject to } -x_1 + 2x_2 &\le\ 8 \\ x_1 + 2x_2 &\le\ 12 \\ 4 \le x_1 \le 10,\ \ x_2 &\ge\ 0 \end{aligned}$$

Standard Form: Free Variables

- Sometimes a variable can be given without any bounds. These types of variables are called free variables.
- To obtain standard form from a free variable, the free variable is expressed as the difference of two non-negative variables. So, if x_i is a free variable, then we get

$$x_i = u_i - v_i, \ 0 \leq u_i, \ 0 \leq v_i.$$

Example

Put the following problem in standard form and write starting basic feasible solution.

$$\begin{aligned} \max z = 10x_1 - 5x_2 - 2x_3 & \\ \textit{subject to } -2x_1 + 3x_2 + 3x_3 &\leq 10 \\ x_1 + x_2 - 5x_3 &= 8 \\ x_1 &\geq -3 \\ -1 \leq x_2 &\leq 1 \\ x_3 & \textit{ free} \end{aligned}$$

Standard Form: Free Variables

Example

$$
\begin{aligned}
\max z &= 10x_1 - 5x_2 - 2x_3 \\
\textit{subject to } -2x_1 + 3x_2 + 3x_3 &\leq 10 \\
x_1 + x_2 - 5x_3 &= 8 \\
x_1 &\geq -3 \\
-1 \leq x_2 &\leq 1 \\
x_3 &\textit{ free}
\end{aligned}
$$