MTM4501-Operations Research

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Week 14

Course Content

- Definition of OR and Its History
- Decision Theory and Models
- Network Analysis
- Inventory Management Models
- Queue Models
    - Waiting Line Models
    - Queuing Theory

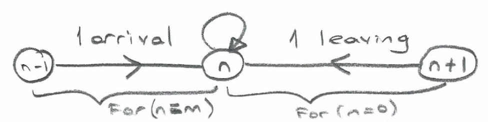
Model 2: Single Source Queue M wIAS & Finite Queue Length

The system can contain at most m customers at any given time. A single server serves a customer, and the queue length cannot exceed $m-1$.

- λ: Arrival rate
- μ: Service rate

Balance Equations:

- For $n=0$: $\lambda P_0 = \mu P_1$
- For $n = 1, 2, ..., m-1$: $\lambda P_{n-1} + \mu P_{n+1} = \lambda P_n + \mu P_n$
- For $n = m$: $\lambda P_{m-1} = \mu P_m$



- For $n=0$: $\lambda P_0 = \mu P_1 \implies P_1 = \frac{\lambda}{\mu} P_0$
- For $n=1$: $\lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1 \implies P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$
- For $n=2$: $\lambda P_1 + \mu P_3 = \lambda P_2 + \mu P_2 \implies P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$

Model 2: Single Source Queue M w/AS & Finite Queue Length

- For $n = m$: $\implies P_m = \frac{\lambda}{\mu} P_{m-1} = \left(\frac{\lambda}{\mu}\right)^m P_0$

In this model, due to the assumption of finite queue length, the sum of probabilities for a finite number of states will be 1. Depending on the values of λ and μ, two cases arise:

- In the case of $\lambda = \mu$:

$$\sum_{n=0}^{m} P_n = 1 \implies \sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n P_0 = (m+1)P_0 = 1 \implies P_0 = \frac{1}{m+1}$$

- In the case of $\lambda \neq \mu$:

$$\sum_{n=0}^{m} P_n = 1 \implies \sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n P_0 = P_0 \frac{1 - \left(\frac{\lambda}{\mu}\right)^{m+1}}{1 - \left(\frac{\lambda}{\mu}\right)} = 1$$

$$\implies P_0 = \frac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{m+1}}$$

Model 2: Single Source Queue M wIAS & Finite Queue Length

Accordingly, the probability of the system being empty can be summarized as follows:

$$P_0 = \begin{cases} \frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{m+1}} & , \lambda \neq \mu \\ \frac{1}{m+1} & , \lambda = \mu \end{cases}$$

The probability of having n customers in the system can be calculated as follows:

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{m+1}} & , \lambda \neq \mu \\ \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{m+1} & , \lambda = \mu \end{cases}$$

L_s: Expected number of customers in the system

$$L_s = \mathbb{E}(n) = \sum_{n=0}^{m} nP_n = \sum_{n=1}^{m} nP_n$$

Model 2: Single Source Queue M w/AS & Finite Queue Length

Again, depending on the values of λ and μ, there are two cases for L_s:

- In the case of $\lambda = \mu$:

$$L_s = \sum_{n=1}^{m} nP_n = \sum_{n=1}^{m} n \cdot \frac{1}{m+1} = \frac{1}{m+1} \sum_{n=1}^{m} n = \frac{1}{m+1} \frac{m(m+1)}{2} = \frac{m}{2}$$

- In the case of $\lambda \neq \mu$:

$$L_s = \sum_{n=1}^{m} nP_n = \sum_{n=1}^{m} n \left(\frac{\lambda}{\mu}\right)^n P_0 = P_0 \underbrace{\sum_{n=1}^{m} n \left(\frac{\lambda}{\mu}\right)^n}_{S_m}$$

$$S_m = 1 \cdot \frac{\lambda}{\mu} + 2 \cdot \left(\frac{\lambda}{\mu}\right)^2 + 3 \cdot \left(\frac{\lambda}{\mu}\right)^3 + \ldots + m \cdot \left(\frac{\lambda}{\mu}\right)^m$$

$$-\frac{\lambda}{\mu} S_m = -\left(\frac{\lambda}{\mu}\right)^2 - 2 \cdot \left(\frac{\lambda}{\mu}\right)^3 - 3 \cdot \left(\frac{\lambda}{\mu}\right)^4 - \ldots - m \cdot \left(\frac{\lambda}{\mu}\right)^{m+1}$$

Model 2: Single Source Queue M wIAS & Finite Queue Length

The last two equations are summed side by side:

$$\begin{aligned}
\left(1-\frac{\lambda}{\mu}\right)S_m &= \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \ldots + \left(\frac{\lambda}{\mu}\right)^m - m\left(\frac{\lambda}{\mu}\right)^{m+1} \\
&= \frac{\lambda}{\mu}\,\frac{1-\left(\frac{\lambda}{\mu}\right)^m}{1-\frac{\lambda}{\mu}} - m\left(\frac{\lambda}{\mu}\right)^{m+1} \\
&= \frac{\frac{\lambda}{\mu}-\left(\frac{\lambda}{\mu}\right)^{m+1}}{1-\frac{\lambda}{\mu}} - (m+1)\left(\frac{\lambda}{\mu}\right)^{m+1} + \left(\frac{\lambda}{\mu}\right)^{m+1} \\
&= \frac{\frac{\lambda}{\mu}-\left(\frac{\lambda}{\mu}\right)^{m+1}+\left(\frac{\lambda}{\mu}\right)^{m+1}-\left(\frac{\lambda}{\mu}\right)^{m+2}}{1-\frac{\lambda}{\mu}} - (m+1)\left(\frac{\lambda}{\mu}\right)^{m+1} \\
&= \frac{\frac{\lambda}{\mu}\left(1-\left(\frac{\lambda}{\mu}\right)^{m+1}\right)}{1-\frac{\lambda}{\mu}} - (m+1)\left(\frac{\lambda}{\mu}\right)^{m+1}
\end{aligned}$$

Model 2: Single Source Queue M wIAS & Finite Queue Length

If the expression is rearranged:

$$S_m = \frac{\frac{\lambda}{\mu}\left(1-\left(\frac{\lambda}{\mu}\right)^{m+1}\right)}{\left(1-\frac{\lambda}{\mu}\right)^2} - \frac{(m+1)\left(\frac{\lambda}{\mu}\right)^{m+1}}{1-\frac{\lambda}{\mu}}$$

Substituting this sum into $L_s = P_0 S_m$:

$$\begin{aligned} L_s =& \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{m+1}} \frac{\frac{\lambda}{\mu}\left(1-\left(\frac{\lambda}{\mu}\right)^{m+1}\right)}{\left(1-\frac{\lambda}{\mu}\right)^2} - \frac{(m+1)\left(\frac{\lambda}{\mu}\right)^{m+1}}{1-\frac{\lambda}{\mu}} \\ =& \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} - \frac{(m+1)\left(\frac{\lambda}{\mu}\right)^{m+1}}{1-\left(\frac{\lambda}{\mu}\right)^{m+1}} \end{aligned}$$

Accordingly, the expected number of customers in the system can be summarized as follows:

$$L_s = \begin{cases} \frac{\lambda}{\mu-\lambda} - (m+1)\frac{\left(\frac{\lambda}{\mu}\right)^{m+1}}{1-\left(\frac{\lambda}{\mu}\right)^{m+1}} & ,\lambda \neq \mu \\ \frac{m}{2} & ,\lambda = \mu \end{cases}$$

Model 2: Single Source Queue M wIAS & Finite Queue Length

- P_m: Probability of the system being busy

$$P_m = 1 - P_0$$

- L_q: Expected number of customers in the queue

$$L_q = L_s - P_m = L_s - (1 - P_0)$$

- λ_e: Effective arrival rate

$$\lambda_e = \lambda(1 - P_m)$$

- W_s: Average waiting time in the system

$$W_s = \frac{L_s}{\lambda_e} = \frac{L_s}{\lambda(1 - P_m)}$$

- W_q: Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda_e} = \frac{L_q}{\lambda(1 - P_m)}$$

Model 3: Infinite Arrival Rate Multi-Server Queue Model

In this model, it is assumed that there are s parallel servers, and each parallel server is identical.

- λ: Arrival rate
- μ: Service rate of each server
- s: Number of parallel servers
- n: Number of customers in the system

The effect of using parallel servers is a proportionate increase in the facility service rate:

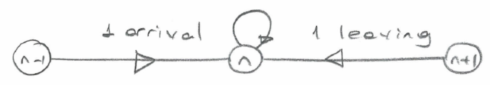
- $n \leq s \implies$ No queue forms
- $n > s \implies$ s customers are in service, and $(n - s)$ customers are waiting in the queue.

In the previous models, it was assumed that $\lambda < \mu$. In this multi-server model, due to s parallel servers, it is assumed that $\lambda < \mu \cdot s$. Here, the product $\mu \cdot s$ can be interpreted as the service capacity.

Model 3: Infinite Arrival Rate Multi-Server Queue Model

Balance Equations: For $0 \leq n < s$;

- For $n = 0, 1, 2, ..., s, \implies \lambda P_{n-1} + (n+1)\mu P_{n+1} = \lambda P_n + n\mu P_n$



- For $n = 0 \implies \lambda P_0 = \mu P_1 \implies P_1 = \frac{\lambda}{\mu} P_0$
- For $n = 1 \implies \lambda P_0 + 2\mu P_2 = \lambda P_1 + \mu P_1 \implies P_2 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 P_0$
- For $n = 2 \implies \lambda P_1 + 3\mu P_3 = \lambda P_2 + 2\mu P_2 \implies P_3 = \frac{1}{3 \cdot 2}\left(\frac{\lambda}{\mu}\right)^3 P_0$
- $\vdots$
- For $n = s \implies P_s = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s P_0$

Model 3: Infinite Arrival Rate Multi-Server Queue Model

Balance Equations: For $s \leq n$;

- For $n = s \implies \lambda P_{s-1} + s\mu P_{s+1} = \lambda P_s + s\mu P_s \implies P_{s+1} = \frac{\lambda}{s\mu} P_s$
- For $n = s+1 \implies \lambda P_s + s\mu P_{s+2} = \lambda P_{s+1} + s\mu P_{s+1} \implies P_{s+2} = \left(\frac{\lambda}{s\mu}\right)^2 P_s$
- $\vdots$
- For $n = s+k \implies P_{s+k} = \left(\frac{\lambda}{s\mu}\right)^k P_s$

By writing $k = n - s$ in the last expression, the probability of having n customers in the system for $s \leq n$ is obtained:

$$P_n = \left(\frac{\lambda}{s\mu}\right)^{n-s} P_s = \left(\frac{\lambda}{s\mu}\right)^{n-s} \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s P_0 = \frac{1}{s!} \frac{1}{s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0$$

For all cases, the probability of having n customers in the system can be summarized as follows:

$$P_n = \begin{cases} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 & , 0 \leq n < s \\ \frac{1}{s!} \frac{1}{s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0 & , s \leq n \end{cases}$$

Model 3: Infinite Arrival Rate Multi-Server Queue Model

Considering the sum of all probabilities:

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{s-1} P_n + \sum_{n=s}^{\infty} P_n = \sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 + \underbrace{\sum_{n=s}^{\infty} \frac{1}{s!} \frac{1}{s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0}_{T}$$

T can be calculated as follows:

$$T = \sum_{n=s}^{\infty} \frac{1}{s!} \frac{1}{s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0 = \sum_{n=s}^{\infty} \frac{1}{s!} \frac{1}{s^{-s}} \left(\frac{\lambda}{s\mu}\right)^n P_0 = \frac{P_0}{s! s^{-s}} \sum_{n=s}^{\infty} \left(\frac{\lambda}{s\mu}\right)^n$$

Under the assumption $\lambda < \mu \cdot s$;

$$T = \frac{P_0}{s! s^{-s}} \left(\frac{\lambda}{s\mu}\right)^s \frac{1}{1 - \frac{\lambda}{s\mu}} = \frac{P_0}{s!} \left(\frac{\lambda}{\mu}\right)^s \frac{1}{1 - \frac{\lambda}{s\mu}}$$

Substituting this into the sum of probabilities;

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 + \frac{P_0}{s!} \left(\frac{\lambda}{\mu}\right)^s \frac{1}{1 - \frac{\lambda}{s\mu}} = 1$$

Model 3: Infinite Arrival Rate Multi-Server Queue Model

Therefore, the probability of the system being empty is calculated as follows:

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \frac{1}{1 - \frac{\lambda}{s\mu}} \right]^{-1}$$

L_q: Expected number of customers in the queue

$$L_q = \mathbb{E}(n - s) = \sum_{n=s}^{\infty} (n - s) P_n = \frac{P_0}{s! s^{-s}} \sum_{n=s}^{\infty} (n - s) \left(\frac{\lambda}{s\mu}\right)^n = \frac{\left(\frac{\lambda}{\mu}\right)^s \frac{\lambda}{s\mu}}{s! \left(1 - \frac{\lambda}{s\mu}\right)^2} P_0$$

L_s: Expected number of customers in the system

$$L_s = L_q + s\frac{\lambda}{s\mu} = L_q + \frac{\lambda}{\mu}$$

Model 3: Infinite Arrival Rate Multi-Server Queue Model

W_s: Average waiting time in the system

$$W_s = \frac{L_s}{\lambda}$$

W_q: Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

Probability of waiting for service

$$\mathbb{P}(n \geq s) = \sum_{n=s}^{\infty} P_n = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left[1 - \frac{\lambda}{s\mu}\right]} P_0.$$

Model 3: Infinite Arrival Rate Multi-Server Queue Model

Example

There are 3 service desks at a post office. Approximately 192 customers arrive every day. Each business day consists of 8 hours. The average service time for each customer is 5 minutes. Therefore;

a) *What is the probability of having no customers in the post office?*

b) *What is the probability of at least one service desk being busy?*

c) *What is the probability of waiting for service?*

d) *What is the expected number of customers in the queue?*

e) *What is the expected number of customers in the system?*

f) *What is the average waiting time for each customer in the queue?*