

Lecture 9

LP Formulation of CPM

PERT Networks

Cost Analysis in Project Management: Crashing the Project

6.5.4 Linear Programming Formulation of CPM

A CPM problem can be thought of as the opposite of the shortest-route problem (Section 6.3), in the sense that we are interested in finding the *longest* route of a unit flow entering at the start node and terminating at the finish node. We can thus apply the shortest route LP formulation in Section 6.3.3 to CPM in the following manner. Define

x_{ij} = Amount of flow in activity (i, j) , for all defined i and j

D_{ij} = Duration of activity (i, j) , for all defined i and j

Thus, the objective function of the linear program becomes

$$\text{Maximize } z = \sum_{\substack{\text{all defined} \\ \text{activities } (i, j)}} D_{ij}x_{ij}$$

(Compare with the shortest route LP formulation in Section 6.3.3 where the objective function is minimized.) For each node, there is one constraint that represents the conservation of flow:

$$\text{Total input flow} = \text{Total output flow}$$

All the variables, x_{ij} , are nonnegative.

Example 6.5-5

The LP formulation of the project of Example 6.5-2 (Figure 6.42) is given below. Note that nodes 1 and 6 are the start and finish nodes, respectively.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	Dummy	<i>G</i>	<i>H</i>	
	x_{12}	x_{13}	x_{23}	x_{24}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	
Maximize $z =$	6	6	3	8	2	11	0	1	12	
Node 1	-1	-1								= -1
Node 2	1		-1	-1						= 0
Node 3		1	1		-1	-1				= 0
Node 4				1			-1	-1		= 0
Node 5					1		1		-1	= 0
Node 6						1		1	1	= 1

The optimum solution is

$$z = 25, x_{12}(A) = 1, x_{24}(D) = 1, x_{45}(\text{Dummy}) = 1, x_{56}(H) = 1, \text{all others} = 0$$

The solution defines the critical path as $A \rightarrow D \rightarrow \text{Dummy} \rightarrow H$, and the duration of the project is 25 days. The LP solution is not complete, because it determines the critical path, but does not provide the data needed to construct the CPM chart.

6.5.5 PERT Networks

PERT differs from CPM in that it bases the duration of an activity on three estimates:

1. **Optimistic time, a** , which occurs when execution goes extremely well.
2. **Most likely time, m** , which occurs when execution is done under normal conditions.
3. **Pessimistic time, b** , which occurs when execution goes extremely poorly.

The range (a, b) encloses all possible estimates of the duration of an activity. The estimate m lies somewhere in the range (a, b) . Based on the estimates, the average duration time, \bar{D} , and variance, v , are approximated as:

$$\bar{D} = \frac{a + 4m + b}{6} \qquad v = \left(\frac{b - a}{6} \right)^2$$

CPM calculations given in Sections 6.5.2 and 6.5.3 may be applied directly, with \bar{D} replacing the single estimate D .

It is possible now to estimate the probability that a node j in the network will occur by a prespecified scheduled time, S_j . Let e_j be the earliest occurrence time of node j . Because the durations of the activities leading from the start node to node j are random variables, e_j also must be a random variable. Assuming that all the activities in the network are statistically independent, we can determine the mean, $E\{e_j\}$, and variance, $\text{var}\{e_j\}$, in the following manner. If there is only one path from the start node to node j , then the mean is the sum of expected durations, \bar{D} , for all the activities along this path and the variance is the sum of the variances, v , of the same activities. On the other hand, if more than one path leads to node j , then it is necessary first to determine the statistical distribution of the duration of the longest path. This problem is rather difficult because it is equivalent to determining the distribution of the maximum of two or more random variables. A simplifying assumption thus calls for computing the mean and variance, $E\{e_j\}$ and $\text{var}\{e_j\}$, as those of the path to node j that has the largest sum of *expected* activity durations. If two or more paths have the same mean, the one with the largest variance is selected because it reflects the most uncertainty and, hence, leads to a more conservative estimate of probabilities.

Once the mean and variance of the path to node j , $E\{e_j\}$ and $\text{var}\{e_j\}$, have been computed, the probability that node j will be realized by a preset time S_j is calculated using the following formula:

$$P\{e_j \leq S_j\} = P\left\{\frac{e_j - E\{e_j\}}{\sqrt{\text{var}\{e_j\}}} \leq \frac{S_j - E\{e_j\}}{\sqrt{\text{var}\{e_j\}}}\right\} = P\{z \leq K_j\}$$

where

z = Standard normal random variable

$$K_j = \frac{S_j - E\{e_j\}}{\sqrt{\text{var}\{e_j\}}}$$

The standard normal variable z has mean 0 and standard deviation 1 (see Section 12.4.4). Justification for the use of the normal distribution is that e_j is the sum of independent random variables. According to the *central limit theorem* (see Section 12.4.4), e_j is approximately normally distributed.

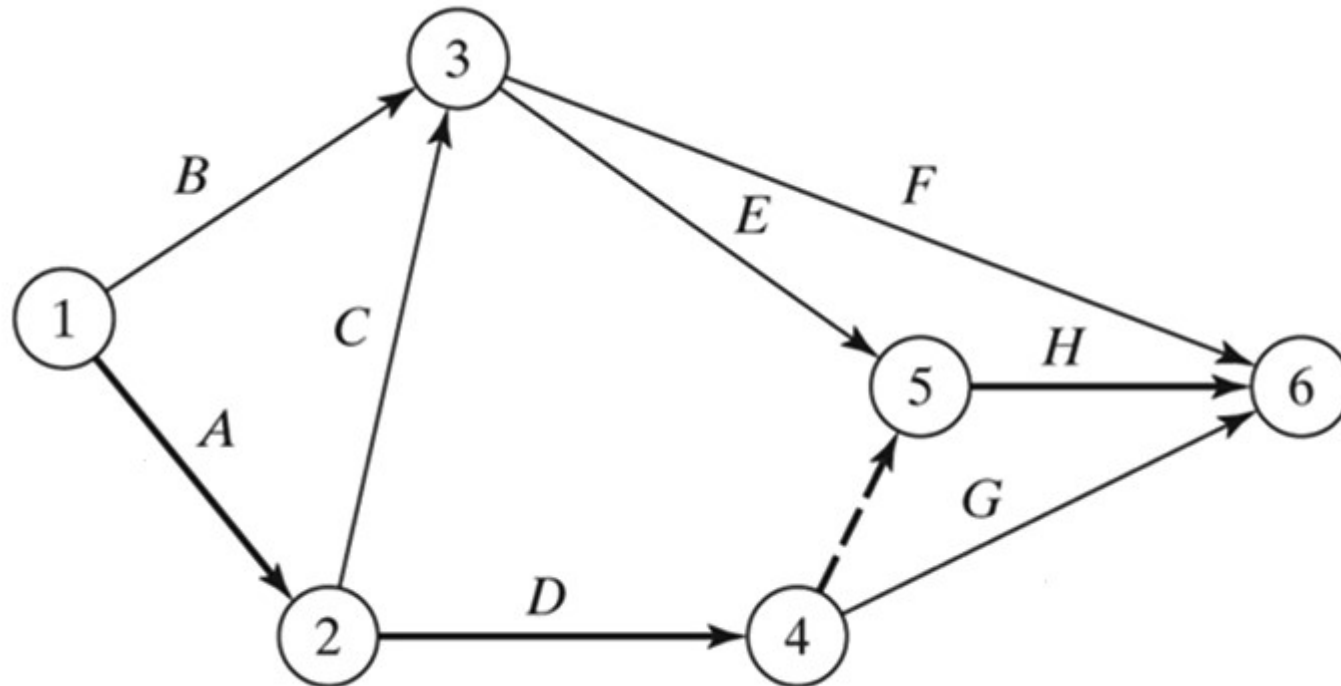
Example 6.5-6

Consider the project of Example 6.5-2. To avoid repeating critical path calculations, the values of a , m , and b in the table below are selected such that $\bar{D}_{ij} = D_{ij}$ for all i and j in Example 6.5-2.

Activity	$i-j$	(a, m, b)	Activity	$i-j$	(a, m, b)
A	1-2	(3, 5, 7)	E	3-5	(1, 2, 3)
B	1-3	(4, 6, 8)	F	3-6	(9, 11, 13)
C	2-3	(1, 3, 5)	G	4-6	(1, 1, 1)
D	2-4	(5, 8, 11)	H	5-6	(10, 12, 14)

The mean \bar{D}_{ij} and variance v_{ij} for the different activities are given in the following table. Note that for a dummy activity $(a, m, b) = (0, 0, 0)$, hence its mean and variance also equal zero.

Activity	$i-j$	\bar{D}_{ij}	V_{ij}	Activity	$i-j$	\bar{D}_{ij}	V_{ij}
A	1-2	5	.444	E	3-5	2	.111
B	1-3	6	.444	F	3-6	11	.444
C	2-3	3	.444	G	4-6	1	.000
D	2-4	8	1.000	H	5-6	12	.444



Copyright © 2011 Pearson Education, Inc. publishing as Prentice Hall

The next table gives the longest path from node 1 to the different nodes, together with their associated mean and standard deviation.

Node	Longest path based on mean durations	Path mean	Path standard deviation
2	1-2	5.00	0.67
3	1-2-3	8.00	0.94
4	1-2-4	13.00	1.20
5	1-2-4-5	13.00	1.20
6	1-2-4-5-6	25.00	1.37

Finally, the following table computes the probability that each node is realized by time S_j specified by the analyst.

Node j	Longest path	Path mean	Path standard deviation	S_j	K_j	$P\{z \leq K_j\}$
2	1-2	5.00	0.67	5.00	0	.5000
3	1-2-3	8.00	0.94	11.00	3.19	.9993
4	1-2-4	13.00	1.20	12.00	-.83	.2033
5	1-2-4-5	13.00	1.20	14.00	.83	.7967
6	1-2-4-5-6	25.00	1.37	26.00	.73	.7673

PROBLEM SET 6.5E

1. Consider Problem 2, Set 6.5b. The estimates (a, m, b) are listed below. Determine the probabilities that the different nodes of the project will be realized without delay.

Project (a)				Project (b)			
Activity	(a, m, b)	Activity	(a, m, b)	Activity	(a, m, b)	Activity	(a, m, b)
1-2	(5, 6, 8)	3-6	(3, 4, 5)	1-2	(1, 3, 4)	3-7	(12, 13, 14)
1-4	(1, 3, 4)	4-6	(4, 8, 10)	1-3	(5, 7, 8)	4-5	(10, 12, 15)
1-5	(2, 4, 5)	4-7	(5, 6, 8)	1-4	(6, 7, 9)	4-7	(8, 10, 12)
2-3	(4, 5, 6)	5-6	(9, 10, 15)	1-6	(1, 2, 3)	5-6	(7, 8, 11)
2-5	(7, 8, 10)	5-7	(4, 6, 8)	2-3	(3, 4, 5)	5-7	(2, 4, 8)
2-6	(8, 9, 13)	6-7	(3, 4, 5)	2-5	(7, 8, 9)	6-7	(5, 6, 7)
3-4	(5, 9, 19)			3-4	(10, 15, 20)		

Cost analysis in project management

Crashing the project

- In many situations, the project manager must complete the project in a time that is less than the length of the critical path.
- So, it is necessary to determine the allocation of resources that minimizes the cost of meeting the project deadline.
- In practice, Linear programming is commonly used while crashing the Project.

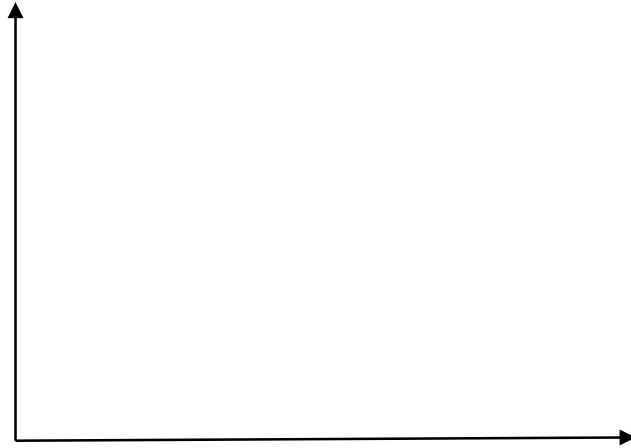
- The parameters:

D_n : Normal completion time

C_n : The cost of completion of activity in normal completion time (D_n)

D_h : Fastest completion time

C_h : The cost of completion of activity in fastest completion time (D_h)



- Suppose that the critical path is determined.
- If we aim to shorten the total duration of project, it is clear that we should select the critical activity or a group of critical activity which has total minimal slope (unit cost).
- The crashing amount is determined by two factors: (1) maximal crashing amount of selected critical activities (2) Free floats of *alternative noncritical paths*.

Alternative noncritical paths

- Starts from the same or backward event with the selected critical activities
- Leads to the critical path(s).

After crashing the project, it is possible to have more than one critical path.

When there doesn't exist any candidate activity to shorten or a critical path is reached its crashing limit, it is not possible to shorten the total project duration, that is crashing process is over.

It should also be noted that a project may have initially more than one critical path.

Example1: Table on below gives the activities of a project, their predecessor(s), durations and costs.

- a) Draw the network for the project.
- b) Determine the critical path.
- c) Construct the Time Schedule.
- d) Obtain programs with minimum costs.

Activities	Predecessor(s)	Durations		Costs	
		Normal	Crashed	Normal	Crashed
A	---	4	2	100	160
B	---	5	5	200	200
C	---	7	4	250	310
D	A	7	5	180	210
E	B	6	6	120	120
F	C	8	4	150	250
G	C	9	6	200	320
H	E, F	4	3	160	220
I	E, F	6	5	240	300
J	E, F, D	10	5	150	350
K	H, G	5	5	250	250

Example2: Table on below gives the activities of a project, their predecessor(s), durations and costs.

- a) Draw the network for the project.
- b) Determine the critical path.
- c) Obtain programs with minimum costs.

Activities	Predecessor(s)	Durations		Costs	
		Normal	Crashed	Normal	Crashed
A	---	5	3	100	150
B	---	4	2	180	200
C	---	6	3	170	200
D	A	7	5	200	260
E	B, C	4	4	120	120
F	B, C	7	7	150	150
G	C	5	3	180	250
H	D, F	8	6	200	300
I	E, G	9	6	100	160 ⁴⁵

References

- [1] *Ahlatçiođlu, M. , Tiryaki, F., Kantitatif Karar Verme Teknikleri, YTÜ Yayın No: YTÜ.FE.DK-98.0349, İstanbul-1998.*
- [2] *H.A.Taha, Operations Research: An Introduction, Prentice Hall; 9th edition, Singapore, 2010.*
- [3] *Winston, W., Operations Research and Applications and Algorithms. 1990.*