Optimization Techniques
Lecture 8

2. Problems with equality and inequality constraints

KKT conditions

3. Problems with nonnegativity constraints

2. Problems with equality and inequality constraints – KKT Multipliers

- Consider the following problem:

$$\text{Min } f(x)$$
$$\text{s.t. } g_i(x)=0 \quad i=1,2,\ldots,k\leq n$$
$$g_j(x)\leq 0 \quad j=k+1,k+2,\ldots,N$$

- where $f(x)$ and $g_i(x)$ are continuously differentiable functions in R^n.

For the above general problem, we adopt the following definitions:

Definition: An inequality constraint $g_j(x) \leq 0$ is said to be **active** at x* if $g_j(x^*) = 0$. It is inactive at x* if $g_j(x) < 0$.
By convention, we consider an equality constraint to be always active.

Definition: Let x* satisfies $g_i(x^*) = 0$ for each *i* and $g_j(x^*) \leq 0$. And let $J(x^*)$ be the index set of active inequality constraints, that is,

$$J(x^*) = \{ j \in \{k+1, \dots, N\} \mid g_j(x^*) = 0 \}$$

We again assumed that the vectors in

$$\{ \nabla g_i, \, i \in \{1, \dots, k\}, \, \nabla g_j, \, j \in J(x^*) \}$$

are linearly independent.

First order necessary condition (Karush – Kuhn – Tucker (KKT) Condition)

- $f, g_i \; and \; g_j$ are continuously differentiable functions in R^n. Let x* be a minimizer for the problem of minimizing f subject to $g_i(x) = 0$ and $g_j(x) \leq 0$. Then the Lagrangian of the problem is:

$$L(x^*, \lambda, \mu) = f(x^*) + \underbrace{\sum_{i=1}^{k} \lambda_i \cdot g_i^*}_{=0} + \underbrace{\sum_{j=k+1}^{N} \mu_j \cdot g_j^*}$$

$$\text{If } \mu_j \cdot g_j = 0 \Rightarrow L(x^*, \lambda, \mu) = f(x^*)$$

$$\mu_j \cdot g_j = 0 \Rightarrow \text{a) } g_j(x^*) = 0.$$

g_j is activ

$$\text{b) } \mu_j = 0 \Rightarrow g_j \leq 0.$$

FONC:

$$\nabla L = \nabla f(x^*) + \sum_{i=1}^{k} \lambda_i \cdot \nabla g_i(x^*) + \sum_{j=k+1}^{N} \mu_j \cdot \nabla g_j(x^*) = 0 \quad \text{---------} (1)$$

$$g_i(x^*) = 0 \quad ; \quad i = 1, \ldots, k \quad \text{----} (2)$$

$$g_j(x^*) \leq 0 \quad ; \quad j = k+1, \ldots N \quad \text{----} (3)$$

$$\mu_j \cdot g_j(x^*) = 0 \quad , \quad j = k+1, \ldots N \quad \text{----} (4)$$

$$\mu_j \geq 0 \quad , \forall j \quad \text{----} (5)$$

Observe that λ_i values are free for all i, but μ_j values are non-negative for all j.

(1) n equations
(2) k "
(3) N − k inequalities.
(4) N − k equalities.
(5) N − k inequalities.

n+N equations.

$x \in \mathbb{R}^n$: n variables.

$\lambda_i \ (i=1,\dots,k)$: k lagrange multipliers

$\mu_j \ (j=k+1,\dots,N)$: N − k KKT multipliers.

We search for all points satisfying (1)-(5), and treat them as candidate extremizers.

Second Order Conditions

We assume that f: $R^n \rightarrow R$ and g: $R^n \rightarrow R^k$ are twice continuously differentiable functions in R^n, the set

$$\{\nabla g_i(x), i = 1,2, \dots, k;\ \nabla g_j(x), j \in J(x^*)\}$$

is linearly independent, the feasible direction set is defined as:

$$\mathrm{H(x)} = \{h \in R^n | \nabla g_i(x)^T h = 0; \forall i = 1,2, \dots, k;\ \nabla g_j(x)^T h = 0;\ \forall j \in J(x^*)\}.$$

Then the second order (sufficient) condition for a local **strict minimizer** is that $\nabla^2 L(x^*, \lambda)$ with respect to x is **positive definite** over H(x*).

Then the SOSC for a local **strict maximizer** is that $\nabla^2 L(x^*, \lambda)$ with respect to x is **negative definite** over H(x*).

Example: Find the extremum points of the below problem.

$$\min f(x) = (x_1 - 1)^2 + (x_2 - 2)^2$$

$$\text{s.t.} \quad \begin{aligned} x_2 - x_1 &= 1 \\ x_1 + x_2 &\leq 2 \end{aligned}$$

3. Problems with nonnegativity constraints

- Consider the following problem:

$$\text{Min } f(x)$$

$$\text{subject to } x_i >= 0 \; ; \; i=1,2,\ldots,n.$$

where f(x) is continuously differentiable functions in R^n. With the following arrangement

$$g_i(x): \quad -x_i <= 0 \; ; \; i=1,2,\ldots,n$$

the above problem can be converted to the following form which is a problem with inequality constraints (Title 2)

$$\text{Min } f(x)$$

$$\text{s.t. } g_i(x) <= 0 \; ; \; i=1,2,\ldots,n$$

- For the above problem, we adapt the FONC and SOSC in title 2:

Observe that $$L(x,\mu) = f(x) + \sum_{i=1}^{n} \mu_i \, g_i(x)$$

FONC:

$$\nabla L = \nabla f + \sum_{i=1}^{n} \mu_i \cdot \nabla g_i = 0 \quad \cdots\cdots (1)$$

$$g_i(x) \leq 0 \quad \cdots\cdots (2)$$

$$\mu_i \cdot g_i(x) = 0 \quad \cdots\cdots (3)$$

$$\mu_i \geq 0 \quad \cdots\cdots (4)$$

$$g_i(x) = -x_i \quad ; \; i = 1, 2, \ldots, n.$$

$$\frac{\partial(-x_i)}{\partial x_i} = -1 \quad ; \; i = 1, 2, \ldots, n$$

$$\frac{\partial(-x_i)}{\partial x_j} = 0 \quad ; \; j = 1, 2, \ldots, n \; ; \; j \neq i$$

$$\frac{\partial \ell}{\partial x_i} = \frac{\partial f}{\partial x_i} - \mu_i = 0 \quad \ldots (1) \quad \forall i.$$

$$\mu_i \cdot g_i(x) = 0 \Rightarrow \mu_i(-x_i) = 0 \Rightarrow \mu_i \cdot x_i = 0 \quad i=1,2,\ldots,n. \quad (2)$$

$$g_i(x) \le 0 \Rightarrow x_i \ge 0 \quad \ldots (3) \quad i=1,2,\ldots,n.$$

$$\mu_i \ge 0 \quad \ldots (4) \quad i=1,2,\ldots,n.$$

We search for points satisfying (1)-(4), and treat these points as candidate extremizers.

(1) $\mu_i = \frac{\partial f}{\partial x_i}$

Considering the above equality, the last form of FONC can be constructed as follows:

FONC:

$$\frac{\partial f}{\partial x_i} \geq 0 \quad ; i = 1, 2, \dots, n \quad \dots\dots (5) \qquad n \text{ inequality}$$

$$x_i \cdot \frac{\partial f}{\partial x_i} = 0 \quad ; i = 1, 2, \dots, n \quad \dots\dots (6) \qquad n \text{ equality}$$

$$x_i \geq 0 \quad ; i = 1, 2, \dots, n \quad \dots\dots (7) \qquad n \text{ inequality}$$

- We have n equations to find n unknowns (n variables x in R^n). Observe that the candidate points must satisfy (5) and (7).

SOSC:

$$\nabla L = \nabla f(x) - \mu_i I. \qquad I = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\nabla^2 L = \nabla^2 f(x) \Longrightarrow$$ L and f have the same Hessian.

$$\nabla^2 L(x^*, \mu) = \nabla^2 f(x^*) = M(x^*).$$

Then the SOSC for a local **strict minimizer** is that $\nabla^2 L(x^*, \mu)$ with respect to x is **positive definite** over H(x*).

Example: Find the extremum points of the below problem.

$$\min f(x, y, z) = 2x^2 + 3y^2 + z^2 - 2xy + 2xz - 2x - 4z$$

s.t. $x \geq 0$, $y \geq 0$, $z \geq 0$

Example: Find the extremum points of the below problem.

$$f(x) = x_1^2 + x_1 x_2 + 2x_2^2 - 6x_1 - 14x_2$$

$$\text{s.t.} \qquad x_1 + x_2 + x_3 = 2$$

$$-x_1 + 2x_2 \le 3$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$$