Optimization Techniques
Lecture 7
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Problems with equality constraints

19.1 INTRODUCTION

In this part, we discuss methods for solving a class of nonlinear constrained optimization problems that can be formulated as:

$$\begin{array}{lll} \text{minimize} & f(\boldsymbol{x}) & \\ \text{subject to} & h_i(\boldsymbol{x}) = 0, & i = 1, \ldots, m \\ & g_j(\boldsymbol{x}) \leq 0, & j = 1, \ldots, p, \end{array}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $h_i : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}$, and $m \leq n$. In vector notation, the problem above can be represented in the following *standard form*:

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \\ & \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}, \end{array}$$

where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$, and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p$. As usual, we adopt the following terminology.

Definition 19.1 Any point satisfying the constraints is called a *feasible point*. The set of all feasible points

$$\{x \in \mathbb{R}^n : h(x) = 0, g(x) \leq 0\}$$

is called the *feasible set*. ■

As we remarked in Part II, there is no loss of generality by considering only minimization problems. For if we are confronted with a maximization problem, it can be easily transformed into the minimization problem by observing that

$$\text{maximize } f(x) = \text{minimize } -f(x).$$

Nonlinear Constrained Optimization

1. Problems with equality constraints
2. Problems with equality and inequality constraints
3. Problems under the nonnegativity constraints

Problems with equality constraints – Lagrange Multipliers

- Consider the following problem:

$$\text{Min } f(x)$$
$$\text{s.t. } g_i(x)=0 \quad i=1,2,\ldots,k\leq n$$

- where $f(x)$ and $g_i(x)$ are continuously differentiable functions in R^n.
- In mathematical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints.
- For the above problem, we introduce new variables λ_i $(i=1,2,\ldots,k)$ called Lagrange multipliers and study the Lagrange function (or Lagrangian) defined by

If x^* is a minimizer of f for the original constrained problem, then there exists λ^* such that (λ^*, x^*) is a stationary point for the Lagrange function

This equality implies that the minimizer of f is also a minimizer of L.

If we apply FONC to L at x^*, we have

If we obtain a minimizer of L, it may be a minimizer of f also. That is, not all stationary points yield a solution of the original problem. Thus, the method of Lagrange multipliers yileds a necessary condition for optimality in constrained problems, but not sufficient.

Finally, Lagrange conditions can be stated as follows:

where the set is linearly independent.

*** A point x* satisfying (1) and (2) need not to be an extremizer.*****

This conditions called first order conditions.

(1)→ k equations X→ n unknowns

(2)→ n equations λ→ k unknowns

Economic Interpretation of lagrange multipliers

- If the RHS 0 of constraint *i* is increased by α_i, then the optimal solution of the problem will depend on α_i also.

The value of the lagrange multiplier at the solution of the problem is equal to the rate of change (contribution) in the value of the objective function as the constrained relaxed.

If $\lambda_i = 0$, then the corresponding constraint is ineffective. The larger λ_i corresponds more effective constraints.

Second Order Conditions

We assume that

- f: $R^n \rightarrow R$ and g: $R^n \rightarrow R^k$ are twice continuously differentiable functions in R^n
- the set is linearly independent
- the feasible direction set is defined as

H(x)=

Then the second order (sufficient) condition for a local minimizer is that

is positive definite over H(x*).

Example: Find the extremum points of the below problem.

$$\min f(x) = {x_1}^2 + {x_2}^2$$
$$\text{s.t. } {x_1}^2 + 2{x_2}^2 - 1 = 0$$

Example: Find the extremum points of the below problem.

$$\min f(x) = x_1 x_2 + 2x_2 x_3 + 3x_1 x_3$$
$$\text{s.t. } x_1 + x_2 + x_3 = 3$$