Midterm 1
MTM5101 - Dynamical Systems and Chaos
Name Surname:
Student ID:

Exam Date: 02/12/2022

QUESTIONS

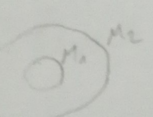
1) (25 pts) For each of the below statements determine whether the claim is true (T) or false (F). No explanation is required.

a) The system $\dot{x}(t) = f(t, x(t), u(t); \beta)$, $t \geq t_0$ is an autonomous system.

b) In dynamical systems $\dot{x}(t) = f(x(t))$, $t \geq t_0$, we don't always have access to the full state x. In other words, we may not measure some elements of x.

c) A dynamical system of the form $\dot{x}(t) = f(x(t))$, $t \geq t_0$ where $x(t) \in \mathbb{R}^3$ may exhibit chaotic behavior. On other words, the system depends too much to the changes in initial conditions.

d) Suppose that, in a dynamical system of the form $\dot{x}(t) = f(x(t), u(t))$, the inputs u_1 and u_2 produces the system states x_1 and x_2, respectively. Then, $u = u_1 + u_2$ will always produce $X = X_1 + X_2$.



e) Suppose that M_1 and M_2 are two closed bounded subsets of $\mathbb{R}^2$ such that int $M_1 \subset$ int M_2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable and all the equilibrium points of $\dot{x} = f(x)$ are in int M_1. Suppose also that there exists a continuously differentiable scalar function $V : \mathbb{R}^2 \to \mathbb{R}$ such that $\langle \nabla V, f \rangle|_{x \in \partial M_1} \leq 0$ and $\langle \nabla V, f \rangle|_{x \in \partial M_2} \leq 0$. Then, $M_2 \setminus M_1$ contains a periodic orbit of the system.

f) Suppose that a system $\dot{x} = f(x)$ has a center, a stable node and a saddle point inside a closed orbit C. Then, there exists a periodic orbit of the system.

g) Suppose that a system $\dot{x} = f(x)$ has a center, a stable node and two saddle points inside a closed orbit C. Then, there does not exist a periodic orbit of the system.

h) Chaos and bifurcations never occur in linear systems.

i) Linear dynamical systems may have limit cycles.

j) Every continuous function is locally Lipschitz.

Here $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the system input and $\beta \in \mathbb{R}^p$ is the vector of system parameters. int M denotes the interior region and ∂M denotes the boundary of a closed bounded subset $M \subset \mathbb{R}^2$.

a	b	c	d	e	f	g	h	i	j
F	T	T	F	F	F	T	T	F	F

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2) Consider the system

$$\dot{x}_1 = ax_1 - x_1x_2$$
$$\dot{x}_2 = bx_1^2 - cx_2$$

where $c > a > 0$. Let $D = \{x \in \mathbb{R}^2 \; : \; x_2 \geq 0\}$.

(a) Let $b > 0$. Show that every trajectory starting in D stays in D for all future time.

(b) Let $b > 0$. Show that there is no periodic orbit through any $x \in D$. *Hint: Use Lemma 2.2.*

(c) Still letting $b > 0$, show that there can be no periodic orbits (in $\mathbb{R}^2$).

(d) This time let $b \leq 0$. Show that there can be no periodic orbits for this case, either.

(a) Let us define $V(x) = x_2 = 0$. The directional derivative along $\partial D = \{x \in \mathbb{R}^2 : x_2 = 0\}$ yields to

$$\langle \nabla V, f \rangle \Big|_{x \in \partial M} = [0 \;\; 1] \begin{bmatrix} a x_1 - x_1 x_2 \\ b x_1^2 - c x_2 \end{bmatrix} \Bigg|_{x_2 = 0} = b x_1^2 - c x_2 \Big|_{x_2=0} = b x_1^2 \geq 0$$

Hence, for every trajectory starting in D stays in D for all future time.

(b) For $x \in D$, we can write

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a - x_2 - c \leq a - c < 0.$$

Since $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is strictly negative in D, by Lemma 2.2 there can be no periodic orbit that entirely lies in D. By part (a), it is not possible that there is a periodic orbit that lies partly in D and partly out of it. Hence there is no periodic orbit through any $x \in D$.

(c) Suppose there is a periodic orbit γ. We know by part (b) that γ must lie entirely in $\mathbb{R}^2 \setminus D$. Corollary 2.1 (the index theorem) tells us that inside γ there should be an equilibrium point, which implies the existence of an equilibrium point in $\mathbb{R}^2 \setminus D$. The system has three equilibria: $(0,0)$, $(\sqrt{ac/b}, a)$, $(-\sqrt{ac/b}, a)$

all of which are in D. Hence γ cannot exist.

(d) For $b \leq 0$, the origin is the only equilibrium point. The linearization at the origin yields to the following A matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

whose eigenvalues are at $\lambda_1 = a > 0$ and $\lambda_2 = -c < 0$. Therefore the origin is a saddle. As a result, by Corollary 2.1, the existence of a periodic orbit is ruled out.

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3) For each of below systems determine whether or not finite escape times occur.

$$\text{(a)} \left\{ \dot{x} = x^2, \right. \qquad \text{(b)} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \text{[redacted]} \ sat(x_1) \end{cases}$$

(a) The unique solution can be obtained as

$$\frac{dx}{x^2} = dt \Rightarrow \int_{x'=x_0}^{x'=x} \frac{dx'}{x'^2} = \int_{t'=t_0}^{t'=t} dt' \Rightarrow -\frac{1}{x'}\Big|_{x'=x_0}^{x'=x} = t'\Big|_{t'=t_0}^{t'=t}$$

$$\Rightarrow \frac{1}{x_0} - \frac{1}{x} = t - t_0$$

$$\Rightarrow x(t) = \frac{x_0}{1 - x_0(t - t_0)}$$

The unique soln exists over $[t_0, \frac{1+x_0 t_0}{x_0})$. As $t \to \frac{1+x_0 t_0}{x_0}$, $x(t)$ leaves any compact set. Therefore, the trajectory has a finite escape time at $t = \frac{1+x_0 t_0}{x_0}$.

(b) The function is not continuously differentiable on $\mathbb{R}^2$. Let us check the Lipschitz property by examining $f(x) - f(y)$. Using $\|\cdot\|_2$ for vectors in $\mathbb{R}^2$ and the fact that the saturation function $sat(\cdot)$ satisfies

$$|sat(x) - sat(y)| \leq |x - y|$$

we obtain

$$\|f(x) - f(y)\|_2^2 \leq (x_2 - y_2)^2 + (x_1 - y_1)^2$$

$$= \left\| \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\|^2 = \|x - y\|^2$$

resulting in a Lipschitz constant $L = 1$. Then, by invoking Theorem 3.2, for any initial condition $x(0)$, the solution $x(t)$ is defined for all $t \in [0, T]$. Since T can be taken arbitrarily, we can conclude that $x(t)$ must exist for all $t \in [0, \infty)$. Hence, the finite escape times are ruled out for this system.

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4) Consider the following system of equations:

$$\dot{x}_1 = x_2 + 3$$
$$\dot{x}_2 = -x_1^2 + 2x_1 - x_2$$

(a) Obtain its linearization at each equilibrium point.
(b) Classify these equilibrium points.
(c) Draw the phase portraits of the linearized system at each equilibrium point.

The equillbrium points of this system are $Q_1(-1,-3)$ and $Q_2(3,-3)$.

(a)&(b) The Jacobian matrix of this system is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -2x_1+2 & -1 \end{bmatrix}$$

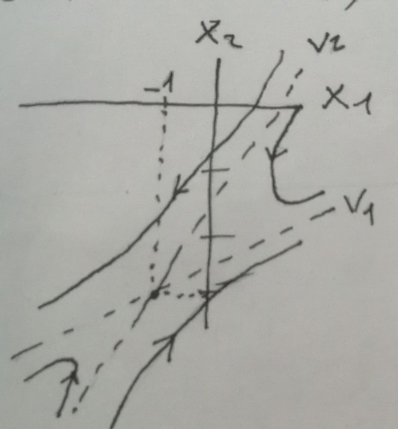
Evaluating the Jacobian matrix at the equilibrium points yields to the following matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$$

Since $\lambda_{1,2}(A_1) = -\frac{1}{2} \pm \frac{\sqrt{17}}{2}$, Q_1 is a saddle point.

Since $\lambda_{1,2}(A_2) = -\frac{1}{2} \pm j\frac{\sqrt{15}}{2}$, Q_2 is a stable focus

(c) For Q_1;



For Q_2;