MTM5101-Dynamical Systems and Chaos

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Week 2

Linear vs Nonlinear Systems

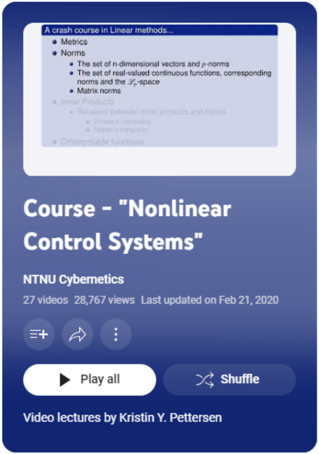


Figure: YouTube Playlist: “Nonlinear Control Systems" Course.

- YouTube Playlist: `https://www.youtube.com/playlist?list=PLc2vvxBHfBcoqxoZXx9wS_3elT-_U0B-r`

Linear Systems

Linear time-invariant (LTI) systems have the following form

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$. The solution of these type of systems will be

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Some control inputs may be

- **proportional state feedback:**

$$u(t) = Kx(t)$$

- **proportional-integral-derivative (PID) feedback:**

$$u(t) = K_p\left(x(t) + \frac{1}{T_i}\int_0^t x(\tau)d\tau + T_d\dot{x}(t)\right)$$

Nonlinear Systems

Nonlinear systems are of the general form

$$\dot{x}(t) = f_p(t, x(t), u(t)),$$
$$u(t) = \gamma(t, x(t))$$

- Generally, no general way to express the analytical solution!
- No linear algebra tools!
- No Laplace transform methods!
- New set of methods for the analysis and design of nonlinear systems is needed.

In this course, we consider nonlinear dynamical systems of the form

$$\begin{aligned} \dot{x}_1 &= f_{p1}(t, x_1, x_2, ..., x_n, u_1, ..., u_m) \\ &\vdots \\ \dot{x}_n &= f_{pn}(t, x_1, x_2, ..., x_n, u_1, ..., u_m) \end{aligned} \tag{1}$$

Note that, the aforementioned system of equations (1) can be written in compact form as

$$\dot{x}(t) = f_p(t, x(t), u(t)), \tag{2}$$

If the system input $u(t) = \gamma(t, x(t))$ is applied to (2) where γ is a function of time t and state x, then we have

$$\dot{x} = f_p(t, x, \gamma(t, x)) = f(t, x) \tag{3}$$

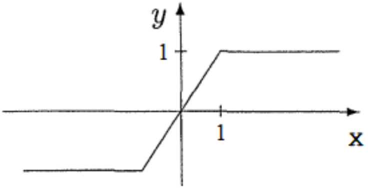
which is called as a TV/nonautonomous system. For the special case when the time t does not appear explicitly in all equations of (1) meaning that the right hand sides are only a function of the state x (not of t), the system is called a TI/autonomous system and can be expressed as $\dot{x} = f(x)$ in compact form.

In this section, we present some examples of specific nonlinear functions and systems:

- **Systems with Input Saturation:** Most actuators (control inputs) have physical limits. So, these type of physical systems can be represented by saturation function as:

$$\begin{aligned} \dot{x} &= Ax + B\,\mathrm{sat}(u) \\ u &= PID(x) \end{aligned} \tag{4}$$

where the saturation function is defined as



$$\mathrm{sat}(x) = \begin{cases} x, & \text{if } |x| \leq 1 \\ \mathrm{sgn}(x), & \text{if } |x| > 1. \end{cases}$$

Figure: Saturation Function $y = \mathrm{sat}(x)$.
(Guess the drawback!)

- **Systems with Input Saturation:** Most actuators (control inputs) have physical limits. For example,
  - valves (musluk vanaları)

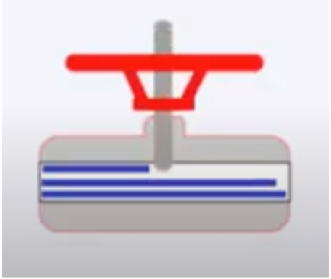


Figure: A Valve.

- **Systems with Input Saturation:** Most actuators (control inputs) have physical limits. For example,
  - variable speed limits
    - max speed limit: 120 km/h
    - min speed limit: 50 km/h

Figure: A VSL Implementation.

- **Systems with Input Saturation:** Alternatively, there is an another type of saturation function in literature which is differentiable and hence enables the linearization of the nonlinear system at an equilibrium point:

$$\mathrm{sat}(x; a, b) = a + \frac{2(b-a)}{\pi} \tan^{-1}(x) \tag{5}$$

where a and b are the minimum and maximum VSL operating values, respectively.

  - **Variable Speed Limit (VSL):** Tool in ITS to improve the measures in traffic flow by changing the speed limit on a highway segment.

    G. Göksu, M. A. Silgu, I. G. Erdagi and H. B. Celikoglu, Integral Input-to-State Stability of Traffic Flow with Variable Speed Limit, *IFAC-PapersOnLine*, 54 (2), 31-36.

    - YouTube Presentation: `https://www.youtube.com/watch?v=a9C8TAf7Zqw`

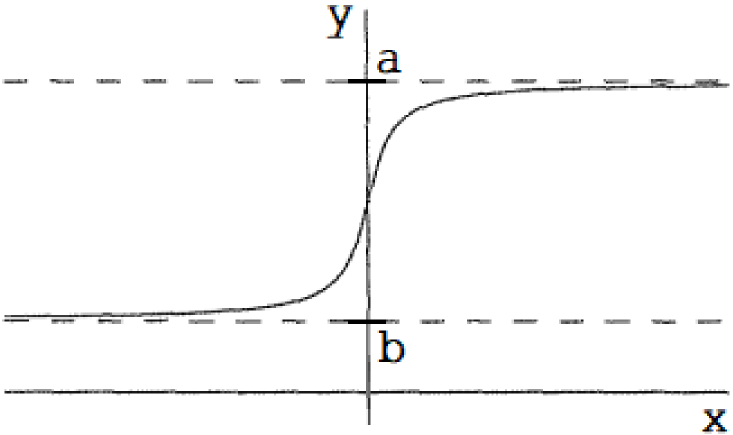


Figure: Saturation Function $y = \text{sat}(x; a, b)$.

- **Systems with Quadratic Terms:** Electric motors in hybrid cars (KYP Video Lectures)
- **Lorentz Attractor:** Lorentz attractor is a system of ordinary differential equations first studied by mathematician and meteorologist Edward Lorenz. It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system. In popular media the “butterfly effect" stems from the real-world implications of the Lorenz attractor, namely that in a chaotic physical system, in the absence of perfect knowledge of the initial conditions (even the minuscule disturbance of the air due to a butterfly flapping its wings), our ability to predict its future course will always fail. This underscores that physical systems can be completely deterministic and yet still be inherently unpredictable.

- **Lorentz Attractor:** The model is a system of three ordinary differential equations now known as the Lorenz equations:

$$\begin{aligned} \dot{x} &= 6(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{6}$$

  Here, the variables of the Lorentz system are given as follows:
  - x: rate of convection,
  - y: horizontal temperature variation,
  - z: vertical temperature variation.

  Even though the model looks like simple, the system behavior of (6) is quite complex and the linear approximation of the system is not able to describe this kind of behavior. In particular, the system displays what is called the “Lorentz attractor", a set of chaotic solutions which resemple a butterfly figure (KYP Video Lecture).

- So, these are some examples where the nonlinearities are so essential that it is not adequate and sometimes not at all possible to describe the system behavior by a linearized model because nonlinear systems may display complex behavior that the linear systems can not reproduce.

Superposition Principle

Linear and nonlinear systems demonstrate different behavior with respect to the linear combination of different inputs.

- **For linear systems $\dot{x} = Ax + Bu$:** If input u_1 produces the state x_1 and the output y_1, and input u_2 produces the state x_2 and the output y_2 then the input $u = u_1 + u_2$ produces $x = x_1 + x_2$ and $y = y_1 + y_2$. Schematically, if

$$u_1 \rightarrow \boxed{\dot{x} = Ax + Bu} \rightarrow y_1 \text{ and } u_2 \rightarrow \boxed{\dot{x} = Ax + Bu} \rightarrow y_2$$

then,

$$u = u_1 + u_2 \rightarrow \boxed{\dot{x} = Ax + Bu} \rightarrow y = y_1 + y_2.$$

- **For nonlinear systems $\dot{x} = f(x, u)$:** There is no such a superposition principle (more complex behavior).

Invariant Sets

When we analyze the dynamics of nonlinear systems, we focus on the behavior in and around invariant sets. This is because we have results saying that if the system states do not escape to infinity, then the system states will always end up in an invariant set. Now, we will analyze the invariant sets of $\dot{x} = f(x)$ (TI systems). One type of invariant sets is the equilibrium point.

Definition (Equilibrium Point)

x^* is an *equilibrium point* of $\dot{x} = f(x)$ if and only if $f(x^*) \equiv 0$ holds.

This definition tells us that if the system starts in the state x^* it will remain in this state for all future time. The general definition definition of invariant sets is as the following.

Definition (Invariant Set)

A set $\mathbb{M}$ is an *invariant set* of $\dot{x} = f(x)$ if and only if

$$x(0) \in \mathbb{M} \Rightarrow x(t) \in \mathbb{M}, \qquad \forall t \geq 0. \tag{7}$$

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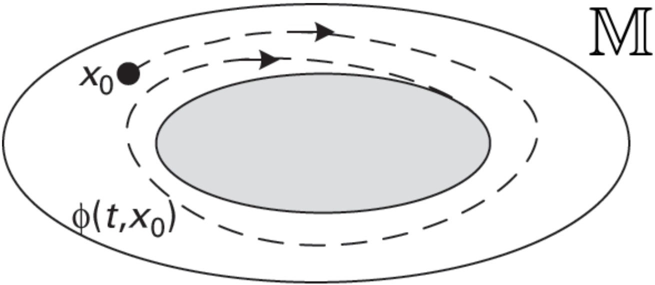
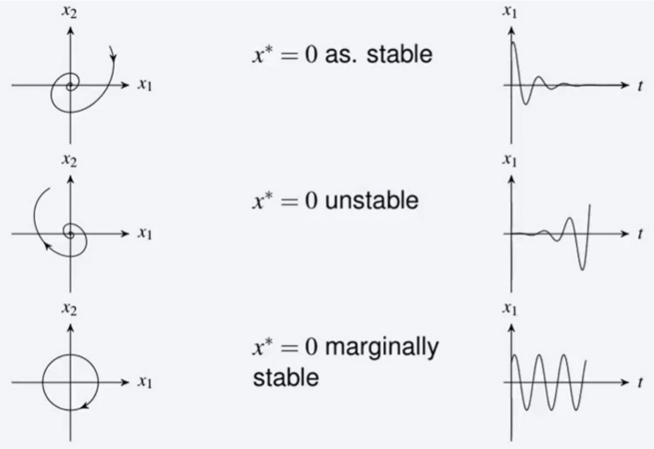


Figure: An Invariant Set Example.

Question: Which invariant sets do linear systems have?

- One equilibrium point: $\dot{x} = Ax \iff x^* \equiv 0$
- May in addition have periodic solutions: $x(t + T) = x(t)$

To illustrate this, let us consider the two dimensional system ($n = 2$).



Question: Which invariant sets do nonlinear systems have?

- Equilibrium points, one or more
- Periodic solutions
  - Limit cycles
- General invariant set (chaos)