MTM5101-Dynamical Systems and Chaos

Gökhan Göksu, PhD

Week 5

Second Order Time-Invariant Systems: Local Analysis

Given the system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

where $f_1, f_2 \in \mathcal{C}^1$ and an isolated equilibrium point $x^* = [x_1^*, x_2^*]^\top$.

- We first linearize the system about the equilibrium point:

$$\Delta\dot{x} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^*, x_2^*) & \frac{\partial f_1}{\partial x_2}(x_1^*, x_2^*) \\ \frac{\partial f_2}{\partial x_1}(x_1^*, x_2^*) & \frac{\partial f_2}{\partial x_2}(x_1^*, x_2^*) \end{bmatrix}}_{=:A} \Delta x \text{ where } \Delta x = \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

- Secondly, we find the eig'val's of the resulting system matrix A: "$\lambda(A)$",
- Finally, we classify the equilibrium point:
  - $\lambda_1, \lambda_2 \in \mathbb{R}$
    - $\lambda_2 < \lambda_1 < 0 \implies$ the equilibrium point is a stable node
    - $0 < \lambda_1 < \lambda_2 \implies$ the equilibrium point is an unstable node
    - $\lambda_2 < 0 < \lambda_1 \implies$ the equilibrium point is a saddle point
  - $\lambda_1, \lambda_2 \in \mathbb{C}$, i.e. $\lambda_{1,2} = \alpha \pm j\beta$, $\alpha \in \mathbb{R}$, $\beta > 0$
    - $\alpha < 0 \implies$ the equilibrium point is a stable focus.
    - $\alpha > 0 \implies$ the equilibrium point is an unstable focus.
    - $\alpha = 0 \implies$ the equilibrium point is a center.

Example

The Jacobian matrix of the function

$$f(x) = \begin{bmatrix} \frac{1}{C}[-h(x_1) + x_2] \\ \frac{1}{L}[-x_1 - Rx_2 + u] \end{bmatrix}$$

with circuit parameters $u = 1.2$ V, $R = 1.5\ k\Omega = 1.5 \times 10^3\ \Omega$, $C = 2\ pF = 2 \times 10^{-12}\ F$ and $L = 5\ \mu H = 5 \times 10^{-6}\ H$ of the tunnel-diode circuit of Example 2.1 in [Khalil, 2002] is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0,5h'(x_1) & 0,5 \\ -0,2 & -0,3 \end{bmatrix}$$

where

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5,$$

$$h'(x_1) = \frac{dh}{dx_1} = 17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4.$$

Example

$$f(x)=\begin{bmatrix}\frac{1}{C}[-h(x_1)+x_2]\\ \frac{1}{L}[-x_1-Rx_2+u]\end{bmatrix} \Longrightarrow \frac{\partial f}{\partial x}=\begin{bmatrix}-0,5h'(x_1) & 0,5\\ -0,2 & -0,3\end{bmatrix}$$

where

$$h(x_1)=17.76x_1-103.79x_1^2+229.62x_1^3-226.31x_1^4+83.72x_1^5,$$

$$h'(x_1)=\frac{dh}{dx_1}=17.76-207.58x_1+688.86x_1^2-905.24x_1^3+418.6x_1^4.$$

Evaluating the Jacobian matrix at the equilibrium points $Q_1=(0,063,0,758)$, $Q_2=(0,285,0,61)$, and $Q_3=(0,884,0,21)$, respectively, yields to

$$A_1=\begin{bmatrix}-3.598 & 0.5\\ -0.2 & -0.3\end{bmatrix},\ \lambda_{1,2}(A_1)=-3.57,-0.33 \quad\Longrightarrow\quad Q_1 \text{ is a stable node,}$$

$$A_2=\begin{bmatrix}1.82 & 0.5\\ -0,2 & -0.3\end{bmatrix},\ \lambda_{1,2}(A_2)=1.77,-0.25 \quad\Longrightarrow\quad Q_2 \text{ is a saddle point,}$$

$$A_3=\begin{bmatrix}-1.427 & 0.5\\ -0,2 & -0.3\end{bmatrix},\ \lambda_{1,2}(A_3)=-1.33,-0.4 \quad\Longrightarrow\quad Q_3 \text{ is a stable node.}$$

Example

The Jacobian matrix of the function

$$f(x) = \begin{bmatrix} x_2 \\ -10\sin\, x_1 - x_2 \end{bmatrix}$$

of the pendulum equation of Example 2.2 in [Khalil, 2002] is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos\, x_1 & -1 \end{bmatrix}$$

Evaluating the Jacobian matrix at the equilibrium points (0,0) and (π,0) yields, respectively, the two matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \ \lambda_{1,2}(A_1) = -0.5 \pm j3.12$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \ \lambda_{1,2}(A_2) = -3.7, 2.7$$

Thus, the equilibrium point $(0, 0)$ is a stable focus and the equilibrium point (π,0) is a saddle point.

Example

The system

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_1 - \mu x_2(x_1^2 + x_2^2)$$

has an equilibrium point at the origin.

The linearized state equation at the origin has eigenvalues $\pm j$. Thus, the origin is a center equilibrium point for the linearized system. We can determine the qualitative behavior of the nonlinear system by representing it in the polar coordinates:

$$x_1 = r\cos\theta \quad \text{and} \quad x_2 = r\sin\theta$$

which yield

$$\dot{r} = -\mu r^3 \quad \text{and} \quad \dot{\theta} = 1$$

From these equations, it can be easily seen that the trajectories of the nonlinear system will resemble a stable focus when $\mu > 0$ and an unstable focus when $\mu < 0$.

Second Order Time-Invariant Systems: Existence of Periodic Orbits

Periodic orbits and limit cycles are **invariant sets** that you may find in phase portraits of nonlinear systems. Linear systems do not have limit cycles!

Definition (Periodic Orbit)

If $\exists\, T > 0$ such that $x(t + T) = x(t)$, $\forall t \geq 0$, then the trajectory $x(t)$ is called a **periodic orbit**.

Definition (Limit Cycle)

A nontrivial (non-constant solutions) isolated (neighbor trajectories are not periodic orbits) periodic orbit (closed trajectory) is called a **limit cycle**.

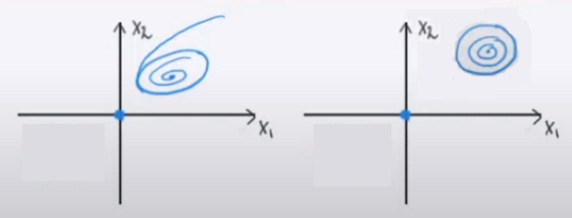


Figure: Isolated vs **Not**-Isolated Periodic Orbits.

There are three existence criteria for periodic orbits:

1. Poincaré - Bendixson criterion
2. Bendixson (negative) criterion
3. Index method

Before, we go further, we present some topological definitions.

Definition (Open set)

A set is an **open set**, if the boundary is excluded from the set

Definition (Closed set)

A set is a **closed set**, if the whole boundary is the part of the set.

Definition (Open ball)

$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$

Definition (Closed ball)

$B_r[x_0] = \{x \in X : d(x, x_0) \leq r\}$

Definition (Bounded set)

A set M is a **bounded set**, if $\exists\, c > 0 : d(x, y) < c, \;\; \forall x, y \in M$.

Now, we are ready to present the first existence criterion for periodic orbits.

Lemma (Poincaré - Bendixson Criterion)

Consider the system

$$\dot{x} = f(x) \quad \text{where} \quad f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ and } f \in \mathcal{C}^1,$$

*Let M be a **closed bounded** (i.e. **compact**) subset of the plane such that*

- *M contains **no** equilibrium points of the system, or it contains only **one** equilibrium point with the property that the eigenvalues of the Jacobian matrix at this point have positive real parts (unstable focus or unstable node).*
- *Every trajectory starting in M stays in M for all future time. In other words, M is a positively invariant set.*

*Then, M contains a **periodic orbit** of the system.*

In order to find an M satisfying these conditions, we will often use closed curves defined by the equation $V(x) = c$ where $c > 0$ and V is some continuously differentiable scalar function ($V : \mathbb{R}^2 \to \mathbb{R}$ and $V \in \mathcal{C}^1$). The choice of the function V decides which kind of curve that $V(x) = c$ describes (eg. circle, ellipse, etc).

Example

Consider the harmonic oscillator given by

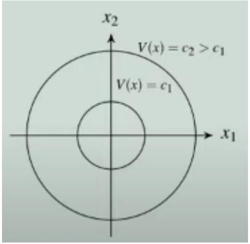
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \lambda_{1,2}(A) = \pm j$$

- What is the equilibrium point of the system? $\Rightarrow Ax = 0 \iff x = (0, 0)$
- Which kind of equilibrium pt. is this? Center, because $Re(\lambda_{1,2}(A)) = 0$

1) The first thing we do is to **choose** a $\mathcal{C}^1$ function $V(x) = c$ to create closed curves which we seek to trap the system trajectories within (PBC Condition 2).

   Let us start with $V(x) = x_1^2 + x_2^2$

$$M = \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$$



Example

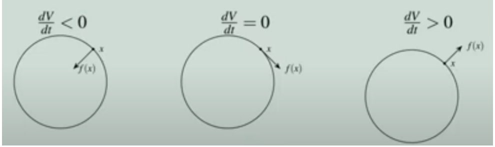
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \lambda_{1,2}(A) = \pm j$$

2) Calculate the directional derivative (Lie derivative)

$$L_f V = \frac{dV}{dt} = \left\langle \frac{\partial V}{\partial x}, \frac{\partial x}{\partial t} \right\rangle = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

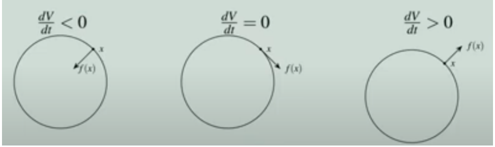
$$= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = 2x_1x_2 - 2x_1x_2 = 0$$

Specifically, we analyze how the function V varies with time when the system is on the curve that we have chosen to define border of M ($\equiv \partial M$):



Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Longrightarrow \quad \lambda_{1,2}(A) = \pm j$$



2) In the border of M, i.e. on ∂M we have

$$\nabla V \cdot f\Big|_{V(x)=c} = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \Bigg|_{V(x)=c} = 0|_{V(x)=c} = 0,$$

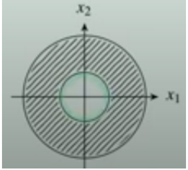
which implies that the trajectories do not leave the set $\{x \in R^2 \mid V(x) \leq c\}$ (PBC Condition 2). But, since this set contains an equilibrium point which is not an unstable focus or node, PBC Condition 1 is not satisfied.

Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ -x_1\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}}_{=:A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \implies \lambda_{1,2}(A) = \pm j$$

2) Let us try an another set for $M = \{x \in \mathbb{R}^2 \mid c_1 \leq V(x) \leq c_2\}$. Since

$$\nabla V \cdot f\Big|_{V(x)=c_1} \geq 0,$$
$$\nabla V \cdot f\Big|_{V(x)=c_2} \leq 0,$$



the trajectories are trapped in M (PBC Condition 2) and there is no equilibrium point, the existence of a periodic orbit in M is guaranteed by invoking Poincaré - Bendixson Lemma.

Warning: Poincaré - Bendixson Lemma ensures existence, not uniqueness!

Lemma (Bendixson (Negative) Criterion)

Consider the system

$$\dot{x} = f(x) \quad \textit{where} \quad f : \mathbb{R}^2 \to \mathbb{R}^2 \textit{ and } f \in \mathcal{C}^1,$$

If, on a simply connected region D on the plane, the expression $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is ***not identically zero*** *and* ***does not change sign****, then the system has* ***no*** *periodic orbits lying entirely in D.*

Let us recall the definition of a **simply connected region**.

Definition (Simply Connected Region)

A region *D* is called a **simply connected region**, if for every simply closed curve C in D, the inner region of C is a subset of D (a curve which does not intersect itself).

Theorem (The Index Theorem)

Let C be a closed curve of the plane and let the indices of the equilibrium points in C are as the following:

- *The index of a node, a focus or a center is $+1$.*
- *The index of a saddle is -1.*

If there exists a periodic orbit $\gamma \in C$, then the sum of the indices of the equilibrium points in C is equal to 1.

The index method is generally useful in ruling out the existence of periodic orbits in certain regions of the plane. As a consequence, if there is no equilibrium point in a closed curve of the plane, then the sum of the indices of the equilibrium points in that closed curve will be different than 1 (note that $p \Rightarrow q \iff q' \Rightarrow p'$).

Theorem (The Index Theorem)

Let C be a closed curve of the plane and let the indices of the equilibrium points in C are as the following:

- *The index of a node, a focus or a center is* $+1$.
- *The index of a saddle is* -1.

If there exists a periodic orbit $\gamma \in C$, then the sum of the indices of the equilibrium points in C is equal to 1.

For instance, if $\sum_i I_i = 0$ where I_i denotes the index of the equilibrium point i, it is guaranteed that there does not exist a periodic orbit in the corresponding closed curve (recall again that $p \Rightarrow q \iff q' \Rightarrow p'$). So, in order to have a periodic orbit, there should exists at least one equilibrium point.

Conversely, in case that $\sum_i I_i = -1 + 1 + 1 = 1$, it is possible to have a periodic orbit in the corresponding closed curve but the index theorem does not guarantee the existence of the periodic orbit.

Lyapunov Stability: Introduction

Starting from next week, we will start introducing

- Lyapunov stability properties for TI (autonomous) systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Local versus global
- Lyapunov stability analysis
  - Lyapunov's indirect method
  - Lyapunov's direct method