MTM5101-Dynamical Systems and Chaos

Gökhan Göksu, PhD

Week 6

Lyapunov Stability: Introduction

Starting from this week, we will start introducing

- Lyapunov stability properties for TI (autonomous) systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Local versus global
- Lyapunov stability analysis
  - Lyapunov's indirect method
  - Lyapunov's direct method

Lyapunov Stability: The Control Problem

- Physical system (a process)
  - Inputs
  - Outputs
- Step of specifications
  - Keep the temperature at a constant value
  - Keep the traffic density at a critical value
  - Keep the concentration of a substance at a certain value
  - Keep the speed of a car at a constant value (cruise control)
- Control law should make the closed-loop system (CLS) according to the following specifications
  - Regulation problem
  - Tracking problem

Lyapunov Stability: Regulation Problem

- $x_{\text{ref}} = \text{const}$
- Ship, car parking, etc.
- Find $u(t) = \gamma(t, x(t))$ such that the closed-loop system (CLS) $\dot{x}(t) = f_p(t, x(t), \gamma(t, x(t))) = \tilde{f}(t, x(t))$ has a desired behavior.
- Desired CLS behavior:

$$\left.\begin{array}{l} - \quad x_{\text{ref}} \text{ an equilibrium point, i.e. } \tilde{f}(t, x_{\text{ref}}) = 0 \\ - \quad \text{Convergence: } \lim_{t\to\infty} x(t) = x_{\text{ref}} \\ - \quad \text{Start close} \implies \text{stay close} \end{array}\right\} \equiv \begin{array}{c}\text{Asymptotic}\\ \text{stability}\end{array}$$

Asymptotic Stabilization Problem

Find $u(t) = \gamma(t, x(t))$ such that x_{ref} is an asymptotically stable equilibrium point of $\dot{x}(t) = f_p(t, x(t), \gamma(t, x(t))) = \tilde{f}(t, x(t))$.

- Coordinate transformation:

$$e(t) = x(t) - x_{\text{ref}} \quad (x = x_{\text{ref}} \iff e = 0),$$

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_{\text{ref}} = \dot{x}(t) = \tilde{f}(t, e(t) + x_{\text{ref}}) = f(t, e(t)) \rightarrow$$ x_{ref} goes as a parameter in the function

Asymptotic Stabilization Problem

Find $u(t) = \gamma(t, e(t))$, $e(t) = x(t) - x_{ref}$ such that $e = 0$ is an asymptotically stable equilibrium point of $\dot{e}(t) = f(t, e(t))$.

Lyapunov Stability: Tracking Problem

- $x_{\text{ref}}(t)$
- Spray painting a car, ship
- Desired CLS behavior:
  - on trajectory $\Longrightarrow$ stay on trajectory
  - convergence to trajectory
  - start close $\Longrightarrow$ stay close
- Coordinate transformation:

$$
\begin{aligned}
e(t) &= x(t) - x_{\text{ref}}(t),\\
\dot{e}(t) &= \dot{x}(t) - \dot{x}_{\text{ref}}(t) = \underbrace{f_p(t, e(t) + x_{\text{ref}}(t), u(t)) - \dot{x}_{\text{ref}}(t)}_{:=\bar{f}_p(t, e(t), u(t))}
\end{aligned}
$$

Asymptotic Stabilization Problem

Find $u(t) = \gamma(t, e(t))$, $e(t) = x(t) - x_{ref}(t)$ such that $e = 0$ is an asymptotically stable equilibrium point of $\dot{e}(t) = \bar{f}_p(t, e(t), \gamma(t, e(t))) = f(t, e(t))$.

Asymptotic Stabilization Problem (Regulation Problem)

Find $u(t) = \gamma(t, e(t))$, $e(t) = x(t) - x_{ref}$ *such that* $e = 0$ *is an asymptotically stable equilibrium point of* $\dot{e}(t) = f(t, e(t))$.

Asymptotic Stabilization Problem (Tracking Problem)

Find $u(t) = \gamma(t, e(t))$, $e(t) = x(t) - x_{ref}(t)$ *such that* $e = 0$ *is an asymptotically stable equilibrium point of* $\dot{e}(t) = f(t, e(t))$.

- Therefore, from now on, for simplicity, we will
  - either consider $\dot{x} = f(t, x)$, $x = 0$ equilibrium point
  - or consider $\dot{x} = f(x)$, $x = 0$ equilibrium point.

In this section, we will start with

$$\dot{x} = f(x), \ f : D \subset \mathbb{R}^n \to \mathbb{R}^n$$

where $x = 0$ is the equilibrium point of this system and f is a locally Lipschitz vector field. We will now define the following Lyapunov stability properties for the equilibrium point of this system:

- Stability
- Asymptotic Stability (AS)
- Exponential Stability (ES)
- Global Asymptotic Stability (GAS)
- Global Exponential Stability (GES)

Definition (Stability)

$x = 0$ is called a **stable** equilibrium point if and only if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \ \forall \geq 0.$$

In this stability definition, it shall always be possible to keep the system state arbitrarily close to the equilibrium point by starting sufficiently close.

Definition (Asymptotic Stability)

The equilibrium point $x = 0$ is (locally) asymptotically stable if and only if

i) $x = 0$ is stable,

ii) $\exists r > 0$ such that $\|x(0)\| < r \Rightarrow \lim_{t \to \infty} x(t) = 0$ (convergence).

Figure: Stability vs Asymptotic Stability.

Stability Definitions and Solution Characteristics

Definition (Asymptotic Stability)

The equilibrium point $x = 0$ is (locally) asymptotically stable if and only if

i) $x = 0$ is stable,

ii) $\exists r > 0$ such that $\|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0$ (convergence).

Definition (Region of Attraction)

The region of attraction R_a (also called region of asymptotic stability, domain of attraction or basin) is the set of all points in $D \subset \mathbb{R}^n$ such that the solution the solution of $\dot{x} = f(x)$, $x(0) = x_0$ is defined for all $t \geq 0$ and converges to the equilibrium point $x = 0$ as $t \to \infty$.

According to this defn., if $R_a = \mathbb{R}^n$, then $x = 0$ is globally asymptotically stable.

Definition (Global Asymptotic Stability)

The equilibrium pt. $x = 0$ is globally asymptotically stable (GAS) if and only if

i) $x = 0$ is stable,

ii) $\forall x(0) \in \mathbb{R}^n$ we have $\lim_{t\to\infty} x(t) = 0$ (global convergence)

GAS guarantees that $x = 0$ is the only equilibrium point!

Stability Definitions and Solution Characteristics

Definition (Global Asymptotic Stability)

The equilibrium pt. $x = 0$ is globally asymptotically stable (GAS) if and only if

i) $x = 0$ is stable,

ii) $\forall x(0) \in \mathbb{R}^n$ we have $\lim_{t\to\infty} x(t) = 0$ (global convergence)

Question: Why there is a need of a requirement of "stability" in the GAS definition? Is convergence not sufficient to imply stability?

The answer of this question is "yes", for linear systems; but "no", for nonlinear systems. So, Convergence $\nRightarrow$ Stability in general. Let us consider the Vinograd's counter example.

Example (Vinograd's Counter-Example)

$$\dot{x} = \frac{x^2(y-x)+y^5}{r^2(1+r^4)}, \quad \dot{y} = \frac{y^2(y-2x)}{r^2(1+r^4)} \quad \textit{where} \quad r^2 = x^2+y^2$$

In this system, the equilibrium point is convergent but it is **not** stable! See the next slide!

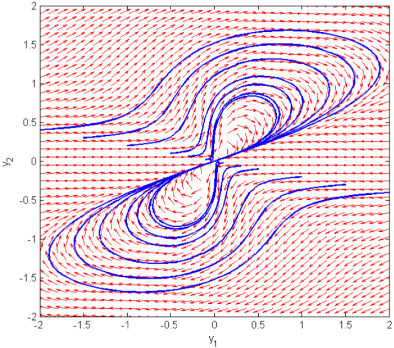

Figure: Vector Field of Vinograd's System.

Figure Credit (See Page 287):

A. Mironchenko, *Input-to-State Stability: Foundations and Applications* Springer Nature, 2023.

See also KYP Lectures (L.3.3-16:07): `https://youtu.be/B5PgJgdlz_Y?list=PLdeo5-jZaFjP9HDqhSt3wzaaVPpRydA9Y&t=967`

Stability Definitions and Solution Characteristics

A particular type of stability exists differing in the way they dissipate along solutions.

Definition (Exponential Stability)

The equilibrium point $x = 0$ is (locally) exponentially stable if and only if $\exists r, k, \lambda > 0$ such that

$$\|x(0)\| < r \Rightarrow \|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall t \geq 0.$$

Again, if $R_A = \mathbb{R}^n$, then $x = 0$ is globally exponentially stable and, again, this also implies that $x = 0$ is the only equilibrium point.

Definition (Globally Exponential Stability)

The equilibrium point $x = 0$ is globally exponentially stable (GES) if and only if $\exists k, \lambda > 0$ such that, $\forall x(0)$, we have

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall t \geq 0.$$

Note that, there is no explicit assumption of stability and convergence because, the estimate guarantee these properties!

Note that, the following implications hold:

- Exponential Convergence $\Rightarrow$ Stability. Because, $\forall \varepsilon > 0$, we can always find $\delta = \frac{\varepsilon}{k}$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| \leq k\|x(0)\| \underbrace{e^{-\lambda t}}_{\leq 1} < k\frac{\varepsilon}{k} = \varepsilon$$

- Exponential Convergence $\Rightarrow$ Asymptotic Stability. Because

$$\lim_{t\to\infty} \|x(t)\| \leq \lim_{t\to\infty} k\|x(0)\|e^{-\lambda t} = k\|x(0)\| \lim_{t\to\infty} e^{-\lambda t} = 0$$

holds for all $\|x(0)\| < \infty$ which implies convergence.

However, the converse of the last implication does not hold in general:

- Asymptotic Stability $\not\Rightarrow$ Exponential Convergence. For example, the system

$$\dot{x} = -x^2, \; x(0) = 1$$

has the solution $x(t) = \frac{1}{1+t}$ which is AS but not ES.

Stability Definitions and Solution Characteristics (Summary)

To sum up, we may summarize the solution characteristics of the Lyapunov stability definitions as the following:

- Stability
- Asymptotic stability $\equiv$ Stability + (local) convergence
- Exponential stability $\equiv$ Stability + (local) exponential convergence
- Global asymptotic stability $\equiv$ Stability + global convergence
- Global exponential stability $\equiv$ Stability + global exponential convergence

Lyapunov Stability Analysis: Lyapunov's Indirect Method

If we have the "explicit" solution of the dynamical system, then we can easily analyze the system behavior. However, this is not the general case. If the solution of the dynamical system can not be obtained "explicitly", then phase plane analysis can be made for second order systems (in $\mathbb{R}^2$). As seen before, the phase portrait of the linearized system demonstrates the same behavior with the phase portrait of nonlinear system, locally. Now, we will demonstrate a new method called Lyapunov's Indirect Method, which is a generalization of the local analysis to $\mathbb{R}^n$.

Theorem (Lyapunov's Indirect Method)

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$ and D is a neighborhood of the origin. Let $\dot{x} = Ax$ be the linearization of this system where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=0}$$

Then,

1) *The origin is asymptotically stable if $Re(\lambda_i(A)) < 0$ for all $i = 1, ..., n$ (in $\mathbb{R}^2$, this corresponds to the equilibrium point being a stable node or stable focus).*
2) *The origin is unstable if $Re(\lambda_i(A)) > 0$ for one or more $i = 1, ..., n$ (in $\mathbb{R}^2$, this corresponds to the equilibrium point being on unstable node, saddle point or unstable focus).*

Note that, in this theorem, we require $\mathcal{C}^1$ not Lipschitz continuity. Note also that, there is no conclusion when $Re(\lambda_i(A)) \leq 0$ for all $i = 1, \ldots, n$ or $Re(\lambda_i(A)) = 0$ for some $i = 1, \ldots, n$. Now, let us see how we can use this theorem to analyze the local stability of a nonlinear dynamical system

Example

Consider the system $\dot{x} = ax - x^3$. Let us analyze the stability properties of the equilibrium point $x = 0$ using Lyapunov's indirect method.

The equilibrium points of this system are $x = 0$, $x = \pm\sqrt{a}$. Since the vector field is $f(x) = ax - x^3$, $f : \mathbb{R} \to \mathbb{R}$ which tells us that $f \in \mathcal{C}^1$, therefore the assumption of Lyapunov's Indirect Method is satisfied. Linearizing about $x = 0$ yields to

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = a - 3x^2\Big|_{x=0} = a \qquad \Rightarrow \qquad \dot{x} = ax$$

and this implies that $\lambda = a$ is the only eigenvalue of the linearized system. According to this result, we have three cases to consider

1) If $a < 0$, then $x = 0$ is an LAS equilibrium point.
2) If $a > 0$, then $x = 0$ is an unstable equilibrium point.
3) If $a = 0$, then the Lyapunov's Indirect Method is inconclusive to classify this equilibrium point $x = 0$. In other words, if the system $\dot{x} = -x^3$, then Lyapunov's indirect method does not give any conclusion for this system.

Corollary (for Lyapunov's Indirect Method)

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x), \; f : D \subset \mathbb{R}^n \to \mathbb{R}^n \quad \text{is} \quad \mathcal{C}^1.$$

Let also $A = \frac{\partial f}{\partial x}\Big|_{x=0}$. Then, the origin is (locally) exponentially stable (LES) if A is Hurwitz (i.e. $Re(\lambda_i(A)) < 0$ for all $i = 1, ..., n$).

In the last example, we, therefore, can conclude that the equilibrium point $x = 0$ of $\dot{x} = ax - x^3$ for $a < 0$ is LES. The equilibrium point $x = 0$ of $\dot{x} = -x^3$ (for $a = 0$) is a LAS but not LES equilibrium point. However, this classification can not be made by using Lyapunov's indirect method and further analysis is required which will be introduced later (Lyapunov's Direct Method).

System Energy and “Energy-Like" Functions

We have learned Lyapunov's indirect method to analyze the (local) stability of the equilibrium point of the systems. In this lecture, we will be able to use Lyapunov's direct method to analyze the stability properties of an equilibrium point which enables us to ensure stability, asymptotic stability and exponential stability of these equilibrium points in “local" and “global" sense.

The motivation for Lyapunov's direct method comes from the consumption of energy of the system. For example, Hamiltonian systems governed by Hamilton's equations.

Consider a two dimensional system is a scalar function of the state and we will denote this by V.

- The energy of the equilibrium point is zero.
- Now, let us draw a curve to all points in the state space where the energy has the same constant value, say c_1. Let us, then draw another curve through all the points in the state space where the energy has another constant where c_2 which is greater than c_1. These curves are called **level curves** or **level surfaces** (in $\mathbb{R}^n$). These curves/surfaces represent constant energy levels for the system.

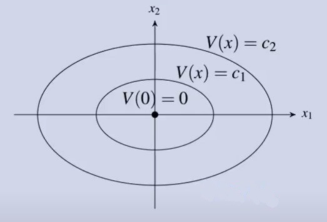

Figure: The Level Surfaces Representing Constant Energy Levels $V(x) = c_i$, $i = 1, 2$ $(0 < c_1 < c_2)$.

Level surfaces $V(x) = c_i$, $0 < c_1 < c_2 < c_3 < \dots$ are the surfaces that represent constant energy levels!

Question: How to choose this energy function to analyze the stability properties of the equilibrium point at the origin?

Recall that $x = 0$ is an equilibrium point of $\dot{x} = f(x)$. What we do is that we study the time evolution of the energy of the system. Specifically, we study the energy evolving along the system trajectories.

System Energy and “Energy-Like" Functions

If the system trajectory moves towards level curves representing higher energy levels, then this corresponds to moving further and further away from the equilibrium point which should suggest that the origin is unstable as shown in the following figure.

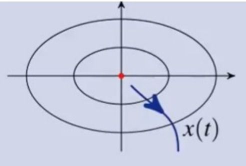

Figure: Energy increases along $x(t)$.

In this case, the time evolution of the energy will be positive:

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) > 0$$

Remember that, we compute the time evolution of the energy along the system trajectories by simply taking the time derivative of the energy function V:

$$\dot{V}(x) = \frac{dV}{dt} = \left\langle \frac{\partial V}{\partial x}, \frac{dx}{dt} \right\rangle = \frac{\partial V}{\partial x} \cdot f(x)$$

Remember also that, this is also called the directional derivative of the function V along the vector field f (or Lie derivative).

System Energy and “Energy-Like" Functions

If the system trajectory moves along the level curves, then this means that the system trajectories intersect curves representing lower and lower energy levels until the energy becomes zero which is at the origin which corresponds to negative time evolution of energy:

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) < 0$$

This indicates that the origin is an asymptotically stable equilibrium point as shown in the following figure.

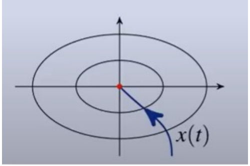

Figure: Energy decreases along $x(t)$.

Note that, we do not need to solve $\dot{x} = f(x)$ in order to see how the energy increases or decreases along $x(t)$. The sign of the directional derivative of the energy function V does this job.

If the time derivative of the energy function is allowed to be zero, then the energy is constant at some future time meaning that the system trajectory moves along one of the level curve. So when the time derivative of the energy function is either negative or 0, mathematically speaking:

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) \leq 0$$

then we do not necesserily have convergence but the behavior is similar to that of a stable equilibrium point as shown in the following figure.

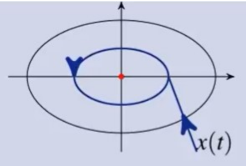

Figure: Energy decreases or is constant along $x(t)$.

These intuitive observations hold for mechanical and electrical systems, which have a well-defined energy concept.

Lyapunov formalized and generalized these intuitive observations for general dynamical systems in his doctoral thesis at Universty of Kharkiv:

A. M. Lyapunov, *The General Problem of the Stability of Motion (In Russian)*, University of Kharkiv Doctoral Dissertation, 1892.

The theorems presented here, which constitute Lyapunov's direct method, are valid for general systems, not only for electrical or mechanical systems for which we have a well-defined energy concept. Instead of the energy function, we must therefore use an “energy-like" function. This “energy-like" function has to serve as a generalized energy function such that it satisfies certain conditions and, therefore, we can use it to analyze stability of the general dynamical systems.