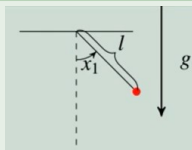


MTM5101-Dynamical Systems and Chaos

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Week 8

Example (Pendulum without Friction)



$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g}{\ell} \sin x_1\end{aligned}$$

Recall the “spoiler” of last week, our Lyapunov function “candidate” may be

$$\begin{aligned}V(x) &= V_{\text{pot}}(x) + V_{\text{kin}}(x) \\ &= -\int_0^{x_1} -\frac{g}{\ell} \sin y \, dy + \frac{1}{2}x_2^2 = \frac{g}{\ell}(1 - \cos x_1) + \frac{1}{2}x_2^2.\end{aligned}$$

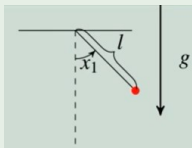
Note that, $V \in C^1$. We choose

$$D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\} \subset \mathbb{R}^2,$$

and this implies

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \in D \setminus \{0\}.$$

Example (Pendulum without Friction)



$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\frac{g}{\ell} \sin x_1$$

Differentiating V along the solutions of the system yields to

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 = \frac{g}{\ell} \sin x_1 x_2 + x_2 \left(-\frac{g}{\ell} \sin x_1 \right) = 0, \quad \forall x \in D$$

This makes sense, since this is a conservative system. Therefore

$$V : D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\} \rightarrow \mathbb{R}$$

is a Lyapunov function for $x = 0$ which tells us that $x = 0$ is a stable equilibrium point.

Example (Pendulum with Friction)

Consider, now, the system governed by pendulum with friction:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell} \sin x_1 - \frac{k}{m} x_2.\end{aligned}$$

For simplicity, let us take $m = 1$.

Since a friction force is acting on this system, the system is no longer a conservative system. Friction is a dissipative force, which draws energy from the system. Let us again choose the same Lyapunov function "candidate", which we know that $V \in C^1$, V is positive definite in $D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\}$. Now, let us check the derivative of V along the solutions of the system:

$$\dot{V}(x) = \frac{g}{\ell} \sin x_1 x_2 + x_2 \left(-\frac{g}{\ell} \sin x_1 - kx_2 \right) = -kx_2^2 \leq 0, \quad \forall x \in D$$

which implies that $x = 0$ is a stable equilibrium point. Moreover, we know that $x = 0$ is an asymptotically stable equilibrium point.

Example (Pendulum with Friction)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{\ell} \sin x_1 - k x_2.$$

Let us replace the term $(1/2)x_2^2$ by more general quadratic form $(1/2)x^T P x$ for some 2×2 positive definite symmetric matrix P :

$$\begin{aligned} V(x) &= \frac{1}{2} x^T P x + \frac{g}{\ell} (1 - \cos x_1) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{g}{\ell} (1 - \cos x_1) \\ &= \frac{1}{2} p_{11} x_1^2 + p_{12} x_1 x_2 + \frac{1}{2} p_{22} x_2^2 + \frac{g}{\ell} (1 - \cos x_1) \end{aligned}$$

For the matrix P to be positive definite, the elements of P must satisfy

$$p_{11} > 0, \quad p_{11} p_{22} - p_{12}^2 > 0$$

The directional derivative of V along the solutions of the system yields to

$$\begin{aligned} \dot{V}(x) &= \left(p_{11} x_1 + p_{12} x_2 \frac{g}{\ell} \sin x_1 \right) x_2 + (p_{12} x_1 + p_{22} x_2) \left(-\frac{g}{\ell} \sin x_1 - k x_2 \right) \\ &= \frac{g}{\ell} (1 - p_{22}) x_2 \sin x_1 - \frac{g}{\ell} p_{12} x_1 \sin x_1 + (p_{11} - p_{12} k) x_1 x_2 + (p_{12} - p_{22} k) x_2^2 \end{aligned}$$

Example (Pendulum with Friction)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell} \sin x_1 - k x_2.\end{aligned}$$

Now, we want to choose p_{11} , p_{12} and p_{22} such that $\dot{V} < 0$. Since the cross-product terms $x_2 \sin x_1$ and $x_1 x_2$ are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = k p_{12}$. With these choices, we have

$$p_{11} p_{22} - p_{12}^2 = p_{12}(k - p_{12}) > 0 \Rightarrow 0 < p_{12} < k \quad (\text{for } k > 0)$$

for $V(x) > 0$. Let us take $p_{12} = \frac{k}{2}$, then $\dot{V}(x)$ will be

$$\dot{V}(x) = -\frac{1}{2} \frac{g}{\ell} k x_1 \sin x_1 - k x_2^2$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Taking $D = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$, we see that $\dot{V}(x) < 0$ over $D \setminus \{0\}$. Thus, by Lyapunov's Direct Method, we can conclude that $x = 0$ is asymptotically stable.

Lyapunov Theorem for Global Asymptotic Stability

Let us consider the system which we analyzed in Lyapunov's indirect method (the linearization method):

$$\dot{x} = -x^3$$

Now, let us analyze the stability properties of the equilibrium point $x = 0$ by using Lyapunov's direct method. The system here may be interpreted as a mechanical system where x is the velocity and a nonlinear friction acts on the system. No potential forces act on the system, so the system energy is the kinetic energy:

$$E = E_{\text{kin}} = \frac{1}{2}v^2 = \frac{1}{2}x^2$$

So, this is one motivation for this choice of Lyapunov function candidate $V(x) = \frac{1}{2}x^2$. An another motivation is that this is a simple choice of a quadratic Lyapunov function candidate $V(x) = \frac{1}{2}x^T P x$ where $P = I$ and since $x \in \mathbb{R}$, we have $V(x) = \frac{1}{2}x^2$.

Note that, $V \in \mathcal{C}^1$, $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$ which implies that V is positive definite in $D \setminus \{0\} = \mathbb{R} \setminus \{0\}$. The directional derivative reads $\dot{V}(x) = x\dot{x} = -x^4 < 0$, $\forall x \neq 0$ which tells us \dot{V} is negative definite in $D \setminus \{0\} = \mathbb{R} \setminus \{0\}$. By Lyapunov's Direct Method, $x = 0$ is LAS. Note that, the conditions for being strict Lyapunov function are satisfied in the whole state space \mathbb{R} , so it is quite natural to ask the following question:

Question: Can we conclude that the origin $x = 0$ is GAS?

Let us consider the following theorem!



Theorem (Lyapunov Theorem for GAS)

If

- \exists a **strict** Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ for $x = 0$ and
- V is radially unbounded

then $x = 0$ is globally asymptotically stable (GAS).

Definition (Radial Unboundedness)

V is **radially unbounded** if and only if $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Example

Turning back to $V(x) = \frac{1}{2}x^2 = \frac{1}{2}\|x\|^2$, this expression tells us that V is a radially unbounded function. This shows that by Lyapunov Theorem for GAS, we can conclude that $x = 0$ is GAS for $\dot{x} = -x^3$.

Question: Why the radial unboundedness condition is necessary to conclude global asymptotic stability based on Lyapunov analysis?

Lyapunov Theorem for Global Asymptotic Stability

For continuously differentiable fcn's, say $V \in \mathcal{C}^1$, the following implications hold

- positive definiteness \Rightarrow level surfaces are closed for small values of c , which is required for local results
- radial unboundedness \Rightarrow level surfaces are closed $\forall c$, which is required for global results

So, if the level surfaces are not closed, we may have that $\|x\| \rightarrow \infty$ even if $\dot{V} < 0$.

Example

Let us take $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$.

Clearly, this function is positive definite. On the other hand,

- For $x_1 = 0, x_2 \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$
- For $x_2 = 0, x_1 \rightarrow \infty \Rightarrow V(x) \rightarrow 1$ as $\|x\| \rightarrow \infty!$

So, $V(x)$ is not radially unbounded. There exist trajectories along which the time derivative of V is strictly negative, meaning that the trajectory intersects level curves corresponding to lower and lower c values, but the trajectory does not converge to the equilibrium point $x = 0$. See the figure on next slide!

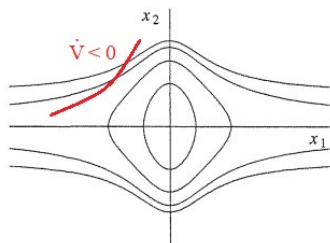


Figure: A Diverging Trajectory with $\dot{V}(x) < 0$.

Although the value of the V function decreases along the trajectory, the trajectory is allowed to slip away from the origin since the level curves are not closed.

See also KYP Lectures (L.4.4-10:57): https://youtu.be/mIkgW_gUKjo?list=PLdeo5-jZaFjNPRGbKxWXrwnkNvjOkP_j8&t=657

Lyapunov Theorem for Global Exponential Stability

We also have a Lyapunov theorem for exponential stability. We still consider the same system as before

$$\dot{x} = f(x)$$

where $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz and $x = 0 \in D$ is an equilibrium point of the system.

Theorem (Exponential Stability)

If there exists a function $V : D \rightarrow \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ such that

- i) $V \in C^1$
- ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a, \forall x \in D$ ($V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$)
- iii) $\dot{V}(x) \leq -k_3 \|x\|^a, \forall x \in D$

Then, $x = 0$ is **exponentially stable (ES)**.

Remark (Global Exponential Stability)

If the conditions in Exponential Stability Theorem are satisfied with $D = \mathbb{R}^n$, then $x = 0$ is globally exponentially stable (GES). The condition (ii) implies radial unboundedness condition. Hence, there is no need to impose radial unboundedness condition for GES.

Some further remarks:

- The Exponential Stability Theorem is also called Barbashin-Krasovskii Theorem.
- $\|\cdot\|$ can be any p -norm on the vector state space.
- This condition is stricter than the Asymptotic Stability Theorem because ES is stricter than AS.

Global Exponential Stability Convergence Rate: If the equilibrium point $x = 0$ of $\dot{x} = f(x)$ is globally exponentially stable, then the solution of the system satisfies

$$\|x(t)\| \leq \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(0)\| e^{-\frac{k_3}{k_2 a} t}, \quad \forall t \geq 0, \quad \|x(0)\| < c$$

where $c > 0$.

Example

Let us analyze the stability properties of the equilibrium point(s) of the system

$$\dot{x} = -x - x^3$$

by using Lyapunov direct method which we analyzed in Lyapunov's indirect method (the linearization method, with $a = 1$).

Note that,

$$\dot{x} = -x - x^3 = -x(1 + x^2) = 0$$

so that $x = 0$ is the only equilibrium point. As shown before, $V(x) = \frac{1}{2}x^2 = \frac{1}{2}\|x\|^2$ is a Lyapunov function candidate for all $x \in \mathbb{R}$ and $V \in \mathcal{C}^1$. (ii) of Exponential Stability Theorem is also satisfied with $k_1 = k_2 = \frac{1}{2}$, $a = 2$. The directional derivative of V along this system reads

$$\dot{V}(x) = x\dot{x} = -x^2 - x^4 \leq -x^2 = -\|x\|^2$$

which tells us that (iii) of Exponential Stability Theorem is satisfied with $k_3 = 1$, $a = 2$. Note that $D = \mathbb{R}$, so that $x = 0$ is GES. The solution of this system satisfies the following GES convergence rate

$$\|x(t)\| \leq \|x(0)\|e^{-t}, \quad \forall t \geq 0.$$