MTM5135-Nonlinear Dynamical Systems

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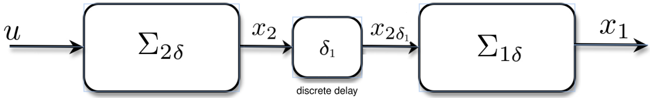
Week 14

Cascade Interconnected iISS TDS



Consider two nonlinear TDS in cascade:

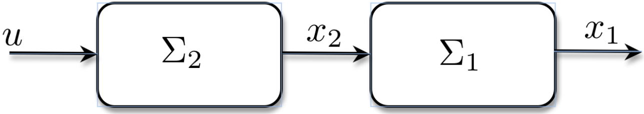
$$\begin{aligned} \Sigma_{1\delta} &: \quad \dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1)) \\ \Sigma_{2\delta} &: \quad \dot{x}_2(t) = f_2(x_{2t}, u(t)) \end{aligned}$$

$\rightarrow \delta_1 \in [0, \delta]$: Interconnection through discrete delay.

Questions:

- iISS preserved under cascade interconnected TDS?
- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and UBEBS?

Cascade Interconnected iISS TDS: Results in Delay-Free Context



Consider two nonlinear systems in cascade:

$$\begin{aligned} \Sigma_1 \ : \quad \dot{x}_1 &= f_1(x_1, x_2) \\ \Sigma_2 \ : \quad \dot{x}_2 &= f_2(x_2, u) \end{aligned}$$

- ISS is naturally preserved in cascade [Sontag, EJC, 1995]
- iISS is not preserved by cascade [Panteley, Loría, Automatica, 2001] & [Arcak et al., SICON, 2002].

Questions:

- ~~iISS preserved under cascade interconnected TDS?~~
- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and UBEBS?

Cascade Interconnected iISS TDS: Results in Delay-Free Context

Theorem [Chaillet, Angeli, SCL, 2008]

Let V_1 and V_2 be two Lyapunov functional candidates. Assume that there exist $\gamma_1, \gamma_2 \in \mathcal{K}$, and $\alpha_1, \alpha_2 \in \mathcal{PD}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $u \in \mathbb{R}^m$,

$$\begin{aligned} \frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) &\leq -\alpha_1(|x_1|) + \gamma_1(|x_2|) \\ \frac{\partial V_2}{\partial x_2} f_2(x_2, u) &\leq -\alpha_2(|x_2|) + \gamma_2(|u|) . \end{aligned}$$

If $\gamma_1(s) = \mathcal{O}_{s\to 0^+}(\alpha_2(s))$, then the cascade is iISS.

$\rightarrow$ $q_2(s) = \mathcal{O}_{s\to 0^+}(q_1(s))$: Given $q_1, q_2 \in \mathcal{PD}$, we say that q_1 *has greater growth* than q_2 around zero if $\exists\, k \geq 0$ such that $\limsup_{s\to 0^+} q_2(s)/q_1(s) \leq k$.

Questions:

- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and BEBS?

Above condition valid for TDS?

Cascade Interconnected iISS TDS: Main Results

Theorem [G, Chaillet, Automatica, 2022]

Assume that $\exists$ two LKF candidates $V_i : \mathcal{C}^{n_i} \to \mathbb{R}_{\geq 0}$, $\sigma_i \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}_\infty$, $i \in \{1, 2\}$, such that, $\forall \phi \in \mathcal{C}^{n_1}$, $v_1 \in \mathbb{R}^{n_2}$

$$D^+ V_1(\phi, v_1) \leq -\sigma_1(|\phi(0)|, \|\phi\|) + \gamma_1(|v_1|), \tag{D1}$$

and, $\forall \varphi \in \mathcal{C}^{n_2}$, $v \in \mathbb{R}^m$

$$D^+ V_2(\varphi, v) \leq -\sigma_2(|\varphi(0)|, \|\varphi\|) + \gamma_2(|v|) \tag{D2}$$

for all $t \geq 0$. Assume further that the following holds:

$$\gamma_1(s) = \mathcal{O}_{s \to 0^+}(\sigma_2(s, 0)). \tag{GR}$$

Then, the cascade is iISS.

Cascade Interconnected iISS TDS: Main Results

Lemma [G, Chaillet, Automatica, 2022]

Let $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ be a LKF candidate satisfying, for all $\phi \in \mathcal{C}^n$,

$$D^+ V(\phi) \leq -\sigma(|\phi(0)|, \|\phi\|),$$

for some $\alpha \in \mathcal{PD}$ and $\eta \in \mathcal{K}_\infty$. Let $\tilde{\alpha} \in \mathcal{PD}$ satisfying

$$\tilde{\alpha}(s) = \mathcal{O}_{s \to 0^+}(\sigma(s, 0)).$$

Then, $\exists$ a continuously differentiable function $\rho \in \mathcal{K}_\infty$ such that the functional $\tilde{V} := \rho \circ V$ satisfies

$$D^+ \tilde{V}(\phi) \leq -\tilde{\alpha}(|\phi(0)|).$$

- Proof can be made applying chain rule to $\tilde{V} := \rho \circ V$.
- Result in finite-dimension: [Sontag, Teel, TAC, 1995]

Cascade Interconnected iISS TDS: Main Results

Proof of Theorem (Sketch).

Proof of Forward Completeness.

- (D2) implies forward completeness of $\dot{x}_2(t) = f_2(x_{2t}, u(t))$.
- (D1) with $u_1(t) = x_2(t - \delta_1) \Rightarrow \nexists$ any finite escape time for $x_1(t)$.

Proof of 0-GAS (Sketch).

- Consider the input-free system
$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t - \delta_1)),$$
$$\dot{x}_2(t) = f_2(x_{2t}, 0).$$
- (GR)+Lemma $\Rightarrow \exists \rho \in \mathcal{K}_\infty \cap \mathcal{C}^1$ such that $\tilde{V}_2 := \rho \circ V_2$ satisfies
$$D^+ \tilde{V}_2(x_{2t}) \leq -2\gamma_1(|x_2(t)|). \tag{1}$$
- Now, consider the LKF defined as
$$\mathcal{V}_2(\phi_2) := \tilde{V}_2(\phi_2) + \int_{-\delta_1}^{0} \gamma_1(|\phi_2(\tau)|)d\tau, \quad \forall \phi_2 \in \mathcal{C}^{n_2}.$$

Proof of Theorem (Continued).

Proof of Theorem: 0-GAS (Sketch-Continued).

- In view of (1), its Dini derivative therefore reads

$$D^{+}\mathcal{V}_2(x_{2t}) \leq -\gamma_1(|x_2(t)|) - \gamma_1(|x_2(t-\delta_1)|). \tag{2}$$

- Furthermore (D1) ensures that

$$D^{+}V_1(x_{1t}, x_2(t-\delta_1)) \leq -\frac{\alpha_1(|x_1(t)|)}{1+\eta_1(V_1(x_{1t}))} + \gamma_1(|x_2(t-\delta_1)|).$$

- Summing this with (2), we get that

$$D^{+}\mathcal{V}(x_t) \leq -\frac{\alpha_1(|x_1(t)|) + \gamma_1(|x_2(t)|)}{1+\eta_1(\mathcal{V}(x_t))},$$

- Replacement of the solution:
  - Point-wise replacement: $\phi(0) \to x(t)$
  - Point-wise replacement: $\phi(-\delta) \to x(t-\delta)$
  - History-wise replacement: $\phi \to x_t$

Proof of Theorem (Continued).

Proof of UBEBS (Sketch).

- (D1)+"Fact" [Angeli et. al., IEEE TAC, 2000] $\Rightarrow \exists \alpha_1 \in \mathcal{K}$, $\eta_1 \in \mathcal{K}_\infty$ such that,
$$D^+ V_1(\phi, v_1) \leq -\frac{\alpha_1(|\phi(0)|)}{1+\eta_1(V_1(\phi))} + \gamma_1(|v_1|), \tag{D1'}$$
- (D2)+"Fact" [Angeli et. al., IEEE TAC, 2000] $\Rightarrow \exists \alpha_2 \in \mathcal{K}$, $\eta_2 \in \mathcal{K}_\infty$ such that,
$$D^+ V_2(\varphi, v_1) \leq -\frac{\alpha_2(|\varphi(0)|)}{1+\eta_2(V_2(\varphi))} + \gamma_2(|v|), \tag{D2'}$$
- (GR)+"Fact" [Angeli et. al., IEEE TAC, 2000] $\Rightarrow \exists k \geq 0$ such that $\alpha_2(s) \geq k\gamma_1(s), \forall s \in [0,1]$.
- Define $\tilde{\eta}_2(s) := s + \eta_2(s) + \frac{k}{\alpha_2(1)}\gamma_1 \circ \underline{\alpha}_2^{-1}(s)(1+\eta_2(s))$, we obtain
$$\frac{\alpha_2(|\varphi(0)|)}{1+\eta_2(V_2(\varphi))} \geq k\frac{\gamma_1(|\varphi(0)||)}{1+\tilde{\eta}_2(V_2(\varphi))}, \quad \forall \varphi \in \mathcal{C}^{n_2}.$$
- Integrating (D1') along the solution of the driving subsystem
$$\int_0^t \gamma_1(|x_2(\tau)|)d\tau \leq \xi_1(\|x_{10}\|) + \xi_2\left(\int_0^t \gamma_2(|u(\tau)|)d\tau\right)$$
for some $\xi_1, \xi_2 \in \mathcal{K}_\infty$ which plays the key role to get UBEBS with $c = 0$.

$\therefore$ 0-GAS + UBEBS with $c = 0 \Rightarrow$ iISS. $\square$

Cascade Interconnected iISS TDS: A Corollary for GAS

Consider the following input-free cascade:

$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1))$$
$$\dot{x}_2(t) = f_2(x_{2t})$$

Corollary [G, Chaillet, Automatica, 2022]

Assume that $\exists$ two LKF candidates $V_1 : \mathcal{C}^{n_1} \to \mathbb{R}_{\geq 0}$ and $V_2 : \mathcal{C}^{n_2} \to \mathbb{R}_{\geq 0}$, $\sigma_1, \sigma_2 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$ such that, $\forall \phi \in \mathcal{C}^{n_1}$, $v_1 \in \mathbb{R}^{n_2}$

$$D^+ V_1(\phi, v_1) \leq -\sigma_1(|\phi(0)|, \|\phi\|) + \gamma_1(|v_1|),$$

and, $\forall \varphi \in \mathcal{C}^{n_2}$,

$$D^+ V_2(\varphi) \leq -\sigma_2(|\phi(0)|, \|\phi\|).$$

Assume also that $\gamma_1(s) = \mathcal{O}_{s \to 0^+}(\sigma_2(s, 0))$. Then, the cascade is GAS.

Example

Consider the following cascade TDS:

$$\dot{x}_1(t) = -\operatorname{sat}(x_1(t)) + \frac{1}{4}\operatorname{sat}(x_1(t-1)) + x_1(t)x_2(t-2)^2 \tag{C1a}$$

$$\dot{x}_2(t) = -\frac{3}{2}x_2(t) + x_2(t-1) + u(t)\int_{t-1}^{t} x_2(\tau)d\tau. \tag{C1b}$$

- $\operatorname{sat}(s) := \operatorname{sign}(s)\min\{|s|, 1\}$ for all $s \in \mathbb{R}$.
- $n_1 = n_2 = 1$, $m = 1$, $\delta_1 = \delta = 2$.

Consider the LKF candidates defined as

$$V_1(\phi_1) := \ln\left(1 + \phi_1(0)^2 + \frac{1}{2}\int_{-1}^{0}\phi_1(\tau)\operatorname{sat}(\phi_1(\tau))d\tau\right),$$

$$V_2(\phi_2) := \ln\left(1 + \phi_2(0)^2 + \int_{-1}^{0}\phi_2(\tau)^2 d\tau\right),$$

Cascade Interconnected iISS TDS: Illustrative Examples

By deriving, we have

$$D^+ V_1(x_{1t}, x_{2t}) \leq -\frac{x_1(t)\mathrm{sat}(x_1(t))}{1+\eta_1(\|x_{1t}\|)} + 2x_2(t-2)^2,$$

$$D^+ V_2(x_{2t}, u(t)) \leq -\frac{x_2(t)^2}{1+\eta_2(\|x_{2t}\|)} + |u(t)|.$$

where $\eta_i(s) = e^{\overline{\alpha}_i(s)} - 1$, $i = 1, 2$. The functions are

- $\alpha_1(s) = \mathrm{sat}(s)s$,
- $\alpha_2(s) = s^2$,
- $\gamma_1(s) = 2s^2$ and
- $\gamma_2(s) = s$.

$\rightarrow$ Growth-rate condition: $2s^2 = \mathcal{O}_{s\to 0^+}(s^2)$.

$\therefore$ Thus, the cascade (C1) is iISS.

Cascade Interconnected iISS TDS: Illustrative Examples

Example

Consider the bilinear TDS in cascade with a discrete-delayed driven and a distributed-delayed driving subsystems

$$\dot{x}(t) = A_1 x(t) + \left(\sum_{i=1}^{n_2} z_i(t) A_{1,i} \right) x(t-\delta) + B_1 z(t), \tag{C2a}$$

$$\dot{z}(t) = A_2 z(t) + \left(\sum_{i=1}^{m} u_i(t) A_{2,i} \right) \int_{-\delta}^{0} z(t+s) ds + B_2 u(t). \tag{C2b}$$

- A_1 and A_2 are Hurwitz matrices.
- All matrices have appropriate dimensions.
- We know from [Pepe, Jiang, SCL, 2006] that (C2a) and (C2b) are iISS but not ISS.

We consider the LKF candidates, for all $\phi \in \mathcal{C}^{n_1}$ and $\varphi \in \mathcal{C}^{n_2}$, as

$$V_1(\phi) = \ln \left(1 + \phi^\top(0) P_1 \phi(0) + p_2 \int_{-\delta}^{0} |\phi(s)|^2 ds \right),$$

$$V_2(\varphi) = \ln \left(1 + \varphi^\top(0) R_1 \varphi(0) + r_2 \int_{-\delta}^{0} \int_{s_1}^{0} |\varphi(s_2)|^2 ds_2 ds_1 \right).$$

where P_1 and R_1 are positive definite symmetric matrices and $p_1 > 0$ and $r_1 > 0$ are scalars to be determined later.

Cascade Interconnected iISS TDS: Illustrative Examples

Example

Consider the bilinear TDS in cascade with a discrete-delayed driven and a distributed-delayed driving subsystems

$$\dot{x}(t) = A_1 x(t) + \left(\sum_{i=1}^{n_2} z_i(t) A_{1,i} \right) x(t-\delta) + B_1 z(t), \tag{C2a}$$

$$\dot{z}(t) = A_2 z(t) + \left(\sum_{i=1}^{m} u_i(t) A_{2,i} \right) \int_{-\delta}^{0} z(t+s) ds + B_2 u(t). \tag{C2b}$$

The derivative of the LKFs along the lines of the subsystems reads

$$D^+ V_1(x_t, z(t)) \leq -\frac{|x(t)|^2}{1 + \eta_1(\|x_t\|)} + \gamma_1 |z(t)|^2$$

$$D^+ V_2(z_t, u(t)) \leq -\frac{|z(t)|^2}{1 + \eta_2(\|z_t\|)} + \gamma_2 |u(t)|^2$$

for $\eta_i(s) = e^{\overline{\alpha}_i(s)} - 1$, $i = 1, 2$ and appropriately chosen $\gamma_1, \gamma_2 > 0$.

$\therefore$ GR condition is also satisfied. Thus, the cascade (C2) is iISS.

Conclusions and Perspectives

Overview:

- Several LK characterizations of iISS for TDS.
- $\mathcal{KL}$ dissipation rates simplify the iISS analysis.
- The existing theory for input/disturbance-free systems is also relaxed.
- Robustness Property: 0-GAS+UBEBS$\Leftrightarrow$iISS.
- Conditions under which the cascade of two iISS TDS is iISS.
- Growth restrictions on the input rate of the driven subsystem and the dissipation rate of the driving one.

Open Questions:

- Converse theorem for iISS LKF with point-wise dissipation.
- Characterizations for ISS TDS.
- Converse theorem for Strong iISS in finite-dimensional context.
- Solns based characterizations for iISS TDS. } C-ISS-TDS
- Conditions to ensure strong iISS for TDS. }
- For further open questions see [Chaillet, Karafyllis, Pepe, Wang, MCSS, 2022, Chapter 8].