MTM5135-Nonlinear Dynamical Systems

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Week 1

Contact Information and Course Evaluation Criteria

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- Midterm: 30 %
- Presentation: 15 %
- Project Report: 15 %
- Final Exam: 40 %

Course Content and Textbooks

- The Anatomy of a Dynamical System
- Nonlinear vs. Linear Systems
- Second Order Time-Invariant Systems
- Lyapunov Stability
- Input-to-State Stability of Time-Invariant Systems

H. K. Khalil, *Nonlinear Systems*, Third Edition, Prentice Hall, 2002.

E. D. Sontag, Input to State Stability: Basic Concepts and Results, *Nonlinear and Optimal Control Theory: Lectures given at the CIME summer school held in Cetraro, Italy*, June 19–29, 2004, pp.163-220.

Dynamical systems: models for the evolving world...

- Car trailer system
  - Video: https://www.youtube.com/watch?v=4jk9H5AB4lM
- Aircraft
  - Video: https://www.youtube.com/watch?v=4UfmsqtTGa0
- Car active suspension system
  - Video: https://www.youtube.com/watch?v=kRt7H0k8A4k
- Building
  - Video: https://www.youtube.com/watch?v=f8lYhpanEVI

A general form of a dynamic system can be given as follows:

$$\dot{x}(t) = \frac{dx}{dt} = f(t, x(t), u(t); \beta), \quad t \geq t_0 \tag{1}$$

where

- $x(t) \in \mathbb{R}^n$ is the state vector,
- $u(t) \in \mathbb{R}^m$ is the system input and
- $\beta \in \mathbb{R}^p$ is the vector of system parameters.

The state vector consists of minimal number of set of values that you need to describe your system and it is a unique minimal description of the system you care about.

The Anatomy of a Dynamical System

System Dynamics and State Space Representation

If the system state and input are the functions of the subsets of the Euclidean space and the evolution of the state vector is described by a differential operator, the dynamical system is called a continuous-time dynamical system. For example, the dynamical system

$$\dot{x}(t) = \frac{dx}{dt} = f(t, x(t), u(t); \beta), \quad t \geq t_0 \tag{1}$$

is a continuous-time system. If the system state and input are the functions of the set of positive integers and the evolution of the state vector is described by a difference operation, the dynamical system is called a discrete-time dynamical system and the general form is given as the following:

$$x(k+1) = f(k, x(k), u(k); \beta), \qquad k = 1, 2, \ldots \tag{2}$$

In this course, we mainly consider continuous-time dynamical systems. The systems of this type are generally defined with a vector valued function that describe the dynamics of the state. For (1), $f : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ tells us, given a current state x and an input u, how does that state change in time. The vector field essentially tells how the derivative is pointing in what vector direction, $\dot{x}$ is pointing at that point in space. In other words, it is a field of these direction vectors.

Dynamical systems can also be classified according to the time-dependency of the vector field. By definition,

- the systems of type $\dot{x} = f(t, x)$ is called a time-varying (TV) or a nonautonomous system whereas
- $\dot{x} = f(x)$ is called a time-invariant (TI) or an autonomous system.

A TV system itself evolves in time and we are actively chaning the system in time. An example can be given as the following.

Example (TV-Car Transmission System)

The simplified dynamics of a car of mass *m* with a transmission gear having a velocity *v* on a road inclined of angle α is

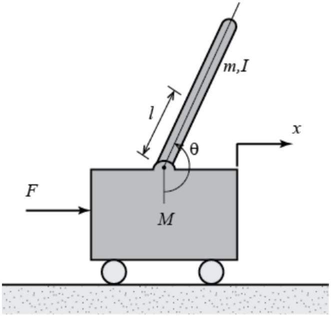
$$\dot{v} = -\frac{k}{m} v^2 sign(v) - g\sin(\alpha) + \frac{G_{\sigma(t)}}{m} T \tag{3}$$

where $G_{\sigma(t)} \in (G_1, G_2, G_3, G_4)$, $G_1 \geq G_2 \geq G_3 \geq G_4$ are the transmission gear ratios, *k* is an appropriate constant and *T* is the torque generated by the engine an input to the model.

A TI system, on the other hand, does not changes over time. An example can be given as the following.

Example (TI-Inverted Pendulum)

An illustration of the inverted pendulum cart is presented as follows:



- F: the control input is the force that moves the cart horizontally
- x: the horizontal position of the cart
- Q: the angular position of the pendulum
- M: mass of the cart
- m: mass of the pendulum
- l: length of the pendulum center of mass

Example (TI-Inverted Pendulum-Cont'd)

The governing equations for the inverted pendulum system are:

$$(M+m)\ddot{x} + ml\ddot{\theta}cos\theta - ml\ddot{\theta}^2 sin\theta = F \tag{4}$$

$$ml^2\ddot{\theta} - mglsin\theta = -ml\ddot{x}cos\theta \tag{5}$$

This system can be represented by θ, $\dot{\theta}$, x and $\dot{x}$.

The Anatomy of a Dynamical System

System Dynamics and State Space Representation

Example (TI-Inverted Pendulum-Cont'd)

Picking $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$ and $x_4 = \dot{x}$, the state space representation of this system will be

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)gsin\theta}{(M+msin^2\theta)l\theta} & -\frac{msin\theta cos\theta\dot{\theta}}{M+msin^2\theta} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mgcos\theta sin\theta}{(M+msin^2\theta)\theta} & \frac{mlsin\theta\dot{\theta}}{M+msin^2\theta} & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{cos\theta}{(M+msin^2\theta)l} \\ 0 \\ \frac{1}{M+msin^2\theta} \end{bmatrix} u \tag{6}$$

The parameters will be $\beta = \begin{bmatrix} M & m & g & l \end{bmatrix}^T$ and, in dynamical systems, changes in these parameters may cause some big changes called “bifurcations".

For practical reasons, you may not always have access to the full state x. For example, in an inverted pendulum example, in case that you have sensors to measure x and θ, you have the corresponding output

$$y = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u. \tag{7}$$

Generally, the output has the form

$$y = g(t, x, u). \tag{8}$$

Here are some modern challenges in dynamical systems:

- **unknown “f":** The systems of the nature have generally a nonlinear nature and the vast majority of them are generally unknown.
- **nonlinear “f":** The systems of the nature are generally nonlinear and this brings out some drawbacks in analysis. For example, superposition principle does not hold for nonlinear systems. That is, if x_1 and x_2 are solutions of a nonlinear system, $x_1 + x_2$ may not be a solution or if u_1 and u_2 are the inputs of a nonlinear system and y_1 and y_2 are the corresponding outputs of the system, $y_1 + y_2$ may not be an output of an input $u_1 + u_2$.
- **high-dimensional:** When high-dimensionality of a system is concerned, it means the high-dimensionality of the state vector x. For example; neurons of brain, weather, neural networks have high amount of state variables which affects the nature and the behavior of the system.

Here are some modern challenges in dynamical systems:

- **multi-scale:** The dynamical systems are not only big but also have a wide range of scales. For example, on the scale of the size of earth there are seasonal changes and on the regional-scale some features also might matter for weather prediction.
- **chaotic:** Not all nonlinear systems are chaotic but many systems are. That is, they are very sensitive to initial conditions (ICs): future depends very strongly to small changes on ICs.
- **latent/hidden vars:** The problem that you might not be measuring the whole state.
- **noise, disturbances**
- **uncertainty**

- **Predict the future:** Dynamical systems are used to predict the future state of the system. According to the nature of the system, the system can be
  - **Deterministic:** No randomness is involved in the development of future states of the system. A deterministic model will thus always produce the same output from a given starting condition or initial state.
  - **Stochastic:** The system may vary in a random manner.
- **Design/Optimization:** I might want to design the system parameters to get a better system performance.
  - **Control:** You can actively manipulate some part of your system in real time.