MTM5135-Nonlinear Dynamical Systems

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Week 3

Common Nonlinear Phenomena - 1

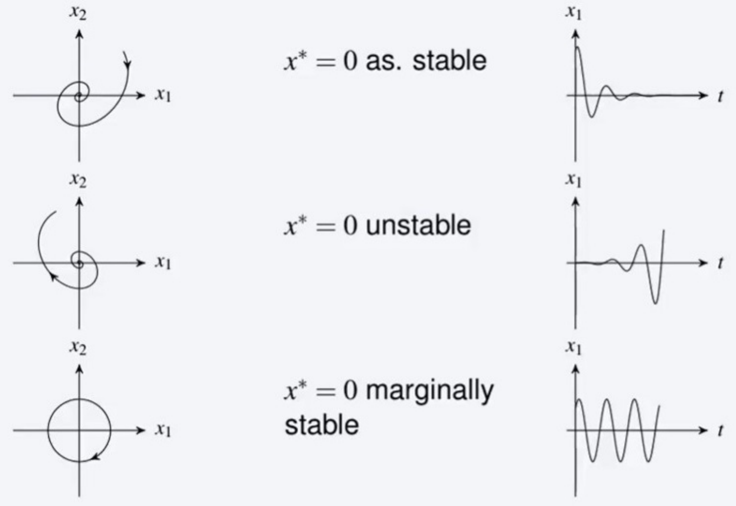
(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)

As a result of the differences between linear and nonlinear systems that we have discussed before there are some phenomena that can be observed in nonlinear dynamical systems which you do not have in linear dynamical systems. As we have seen, the behavior, that the nonlinear system displays, depends on the ICs and as a result of this, in nonlinear control systems, the stability that the system displays may depend on the magnitude of the reference signal given as an exogeneous input to the system.

Linear systems had asymptotically stable, unstable or marginally stable behavior which of these behaviors did not depend on ICs whether the system started close to or far away from the equilibrium point. In other words, for linear systems, there was no difference between the local and the global behavior when considering the type of stability. Example: Motor valve process (KYP Video Lecture).

Common Nonlinear Phenomena - 1

(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)



Common Nonlinear Phenomena - 1

(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)

In linear systems, you may obtain oscillations in two ways:

- If you perturb the linear system by a sinusoidal input, then the system state/output will oscillate as well. For example, let us consider the following scalar linear equation

  $$\dot{x}(t) = bu(t), \qquad u = \sin(\omega t), \qquad t \geq 0.$$

  The solution of this equation can be expressed as

  $$\begin{aligned} x(t) &= x_0 + b \int_{\tau=0}^{\tau=t} \sin(\omega\tau) d\tau \\ &= x_0 + (1/\omega)B - (1/\omega)B\cos(\omega t) \end{aligned}$$

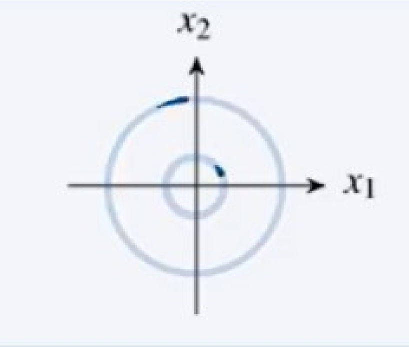
  which has an oscillatory behavior.

Common Nonlinear Phenomena - 1

(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)

In linear systems, you may obtain oscillations in two ways:

- We can also have periodic solutions in linear systems without any external input. When the system is marginally stable, we get periodic solutions. Note that, the amplitude of these solutions is given by the initial value.

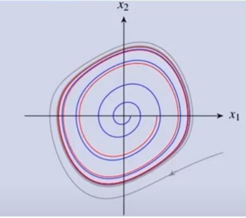


Common Nonlinear Phenomena - 1

(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)

Another phenomenon that you may observe in nonlinear systems which you do not have in linear systems is that of a stable periodic solution. These solutions are called limit cycles.

For such kind of systems that you may observe limit cycles, differently from linear systems, the frequency and the amplitude of the solutions may be independent of the initial state and in the case when the limit cycle is stable, the trajectories surrounding it converge to the limit cycle.



The same also happens when it starts inside the limit cycle. This is something that we can never get from a linear system. Therefore, in practice, if we want to create stable oscillations, we must use a nonlinear system.

One well-known example such an oscillator is described by the Van der Pol oscillator which was proposed by the Dutch electrical engineer and physicist Balthazar Van der Pol while he was working for Philips.

Example (Van der Pol Oscillator)

The Van der Pol oscillator is described by the following equation:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0.$$

This equation describes a circuit with a resistance, an inductor and a capacitor where the resistance element is nonlinear.[1]

[1] For more details, see: *scholarpedia.org/Van_der_Pol_oscillator*

Common Nonlinear Phenomena - 1

(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)

Example (Van der Pol Oscillator)

The Van der Pol oscillator is described by the following equation:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0.$$

An intuitive description why this circuit produces stable oscillations is that depending on the value of x, the sign of the resistor coefficient "$\varepsilon(x^2 - 1)$" changes.

- When $x > 1$ the sign of the coefficient is positive and the resistor element, thus consumes energy.
- When $x < 1$, the sign of the resistor coefficient is negative and the resistor element generates energy.

This transition between alternately consuming and producing energy generates stable oscillations.

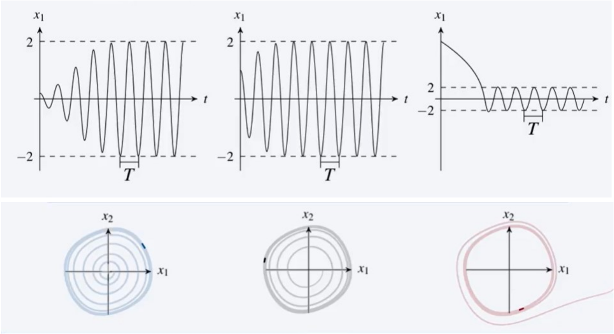
Common Nonlinear Phenomena - 1

(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)

Example (Van der Pol Oscillator)

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0.$$

The simulation results with $\varepsilon = 0.1$ shows that the frequency and the amplitude of the time evalution of x does not depend on ICs!



Common Nonlinear Phenomena - 1

(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)

Other phenomena that you may have in nonlinear systems which never occur in linear systems are chaos and bifurcations. Bifurcation means that the system behavior changes character with only small changes in the parameters of the system while for the systems with chaos the system behavior changes character with small changes in the initial value. For example, let us consider the following example

Example

Consider a second order system perturbed by a sinusoidal input:

$$\ddot{x} + 0.1\dot{x} + x^5 = 6 \sin t$$

Let us also take the following two different ICs:

- $x(0) = 2,\ \dot{x}(0) = 3$
- $x(0) = 2.01,\ \dot{x}(0) = 3.01$

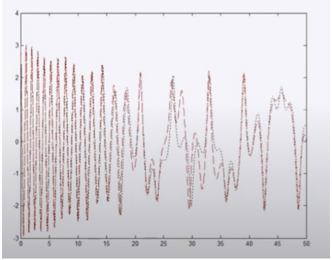
Common Nonlinear Phenomena - 1

(*Tür:* Yaygın Görülen Bazı Nonlineer Sistem Özellikleri)

Example

- $\ddot{x} + 0.1\dot{x} + x^5 = 6 \sin t$
- - - - $x(0) = 2$, $\dot{x}(0) = 3$
- - - - $x(0) = 2.01$, $\dot{x}(0) = 3.01$

Even the initial values are almost the same, we see that the behavior of the state is different in the two cases. We may expect a sinusoidal output in steady state, but the equation has nonlinearity x^5 and because of this simple nonlinearity, the system demonstrates chaos behavior. (KYP Video Lecture)

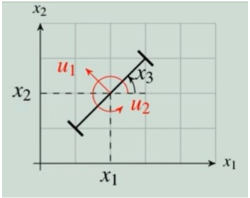


Common Nonlinear Phenomena - 2

The Trivial Question: When to Use NL Analysis/Design?

An important motivation may be that if you see that the system displays behavior that is recognized as nonlinear phenomena that do not occur in linear systems, then you know that the system cannot be properly described by a linear model. If this behavior is essential, meaning that it does not only occur in extreme conditions but also in a particular region of the state space where the system will be operating (for instance, close to the invariant set that you are interested in stabilizing), then you should use a nonlinear model and the corresponding nonlinear analysis and control design tools.

Example (Car Parallel Parking)



- Kinematic model:

$$\dot{x}_1 = \sin(x_3)u_1$$

$$\dot{x}_2 = \cos(x_3)u_1$$

$$\dot{x}_3 = u_2$$

x_1, x_2: position of the center point of the front axle,

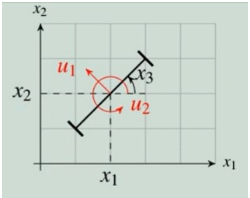
x_3 : angle of the front axle with respect to x_1 -axis,

u_1 : forward velocity

u_2 : angular velocity

- Control problem: Parallel parking, i.e. find $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ that makes the system asymptotically stable (we want all $x_i \to 0$).

Example (Car Parallel Parking)



- Kinematic model:

  $\dot{x}_1 = \sin(x_3)u_1$

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- Control problem: Parallel parking, i.e. find $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ that makes the system asymptotically stable (we want all $x_i \to 0$).

- Linearized model: Using $\sin(x_3) \approx 0$ and $\cos(x_3) \approx 1$

$$\left.\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= u_2 \end{aligned}\right\} \text{Not controllable!} \implies \text{Cannot affect car's position along } x_1\text{-axis!}$$

- So, this is also a case where a linear model does not properly describe the real system!

Fundamental Properties: Existence and Uniqueness

For a system $\dot{x} = f(t, x)$, we expect that starting an experiment from a certain initial state at time $t = t_0$, i.e. $x(t_0) = x_0$, the system will move and its state will be well-defined up to a future time. In addition, for a deterministic system we expect that if we repeat the experiment with exactly the same initial state at time t_0, then we would get exactly the same motion.

Theorem (Local Existence and Uniqueness)

If
- *$f(t, x)$ is piecewise continuous in t,*
- *$f(t, x)$ satisfies*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|, \ \forall x, y \in B, \ \forall t \in [t_0, t_1] \qquad \text{(LC)}$$

where the ball B is defined as $B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$.

Then, there exists a unique solution of the initial value problem (IVP) $\dot{x} = f(t,x), x(t_0) = x_0$, i.e. $x(t)$ is defined locally on $t \in [t_0, t_0 + \delta]$ for some $\delta > 0$.

- LC: Lipschitz Condition

Fundamental Properties: Existence and Uniqueness

If $f : \mathbb{R} \to \mathbb{R}$ and we rearrange the Lipschitz condition a little, then we have

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L \tag{LC-$\mathbb{R}$}$$

So, the Lipschitz condition in this case says that the slope cannot be infinitely large and it has to be bounded. In other words, the Lipschitz condution says that the function should not grow too fast.

Geometrical Interpretation: *Google Drive (Lipschitz.mp4)*

Example

Let us consider the dynamical system $\dot{x} = \sqrt[3]{x}$ with the IC $x(0) = 0$. Note that, the vector field $f(x) = \sqrt[3]{x}$ is defined as $f : \mathbb{R} \to \mathbb{R}$ and, therefore, we may use the condition (LC-$\mathbb{R}$). Differentiating f yields to

$$\left|\frac{df}{dx}\right| = \left|\frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}\right| \underset{x \to 0}{\to} \infty. \tag{1}$$

So, the function $f(x) = \sqrt[3]{x}$ has an infinite slope at zero and, therefore, it does not satisfy the Lipschitz condition of Theorem 8. So, Theorem 8 does not guarantee the existence and uniqueness of the solutions of the IVP. Geometrically, we can not “trap" the function $f(x) = \sqrt[3]{x}$ by the lines with slopes L and $-L$ around $x = 0$. However, the IVP has actually solutions. It is possible to solve IVP analytically and we can prove that there exists a solution but is not unique:

- $x(t) = \sqrt[\frac{3}{2}]{\frac{2}{3}t}$ and $x(t) = 0$ are solns of this IVP.

Fundamental Properties: Existence and Uniqueness

The key assumption of the “Local Existence and Uniqueness Theorem" was “locally Lipschitzness". There are some further related definitions of this property.

Definition

The function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is said to be

- locally Lipschitz on $D \subset \mathbb{R}^n$, if f is locally Lipschitz at any point in D. The Lipschitz constant L can be different for each point and can vary over D,
- Lipschitz on D, if L is same for all points in D,
- globally Lipschitz, if f is Lipschitz on the whole state space $\mathbb{R}^n$.

Based on these definitions, we have these chain of inclusions for different types of Lipschitz continuity:

$$\text{locally Lipschitz on } D \subset \text{Lipschitz on } D \subset \text{globally Lipschitz}$$

- Domain: Open and Connected Set!

Fundamental Properties: Existence and Uniqueness

As seen in the previous example, if a function fails to be discontinuous at a certain point of the subset D, then it also fails to satisfy locally Lipschitzness at the point of discontinuity. On contrary, the locally Lipschitzness can be guaranteed by the boundedness of partial derivatives which is demonstrated in the following lemma.

Lemma (Bounded Derivatives $\Longrightarrow$ Locally Lipschitzness)

Let $f : [a,b] \times D \to \mathbb{R}^n$ be continuous for some domain $D \subset \mathbb{R}^n$. Suppose that $\left[\frac{\partial f}{\partial x}\right]$ exists and is continuous on $[a,b] \times D$. If, for a convex subset $W \subset D$, there is a constant $L \geq 0$ such that

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq L \text{ on } [a,b] \times W,$$

then,

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \quad \text{for all } t \in [a,b], \quad x, y \in W.$$

- Recall: Convex Set!

Fundamental Properties: Existence and Uniqueness

We have the following lemma, if the uniform boundedness of the partial derivatives is not satisfied.

Lemma (Continuous Derivatives $\Longrightarrow$ Locally Lipschitzness)

Let $f : [a, b] \times D \to \mathbb{R}^n$ be continuous for some domain $D \subset \mathbb{R}^n$. Suppose that $\left[\frac{\partial f}{\partial x}\right]$ exists and is continuous on $[a, b] \times D$. Then f is locally Lipschitz on D.

Lemma (Cont.&Cont. Der's. on $\mathbb{R}^n$ $\Longrightarrow$ Gl. L'ness. $\Longleftrightarrow$ Bdd. Der's.)

If f(t,x) and $\left[\frac{\partial f}{\partial x}\right](t, x)$ are continuous on $[a, b] \times \mathbb{R}^n$, then f is globally Lipschitz on $[a, b] \times \mathbb{R}^n$ if and only if $\left[\frac{\partial f}{\partial x}\right]$ is uniformly bounded on $[a, b] \times \mathbb{R}^n$.

Fundamental Properties: Existence and Uniqueness

Example

The function

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

is continuouly differentiable on $\mathbb{R}^2$.

- It is not globally Lipschitz since $\left[\frac{\partial f}{\partial x}\right](t,x)$ is not uniformly bounded on $\mathbb{R}^2$ (i.e. is depending on x).
- However, on any compact subset of $\mathbb{R}^2$, f is Lipschitz.

Suppose $W = \{x \in \mathbb{R}^2 \quad |x_1| \leq a_1, \quad |x_2| \leq a_2\}$. The Jacobian matrix is given by

$$\left[\frac{\partial f}{\partial x}\right](t,x) = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}.$$

Fundamental Properties: Existence and Uniqueness

Example

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

Using the induced matrix norm ($\infty - norm$), we have

$$\left\| \frac{\partial f}{\partial x} \right\|_\infty = \max\{|-1 + x_2| + |x_1|, |x_2| + |1 - x_1|\}$$

All points in W satisfy

$$|-1 + x_2| + |x_1| \leq 1 + a_1 + a_2 \; and \; |x_2| + |1 - x_1| \leq 1 + a_1 + a_2$$

Hence, we have

$$\left\| \frac{\partial f}{\partial x} \right\|_\infty \leq 1 + a_1 + a_2.$$

By “Bounded Derivatives $\Longrightarrow$ Locally Lipschitzness Lemma", we can conclude that f satisfies the following inequality

$$\|f(x) - f(y)\| \leq (1 + a_1 + a_2)\|x - y\| \quad x, y \in W$$

and a Lipschitz constant can be taken as $L = 1 + a_1 + a_2$.

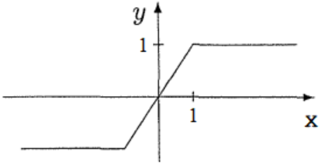
Fundamental Properties: Existence and Uniqueness

Example

Next week, we will continue the Lipschitz continuity of the following function

$$f(x) = \begin{bmatrix} x_2 \\ -\mathrm{sat}(x_1 + x_2) \end{bmatrix}$$

where the saturation function is defined as



$$\mathrm{sat}(x) = \begin{cases} x, & \text{if } |x| \leq 1 \\ \mathrm{sgn}(x), & \text{if } |x| > 1. \end{cases}$$

Let us see, what will happen ☺