MTM5135-Nonlinear Dynamical Systems

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Week 4

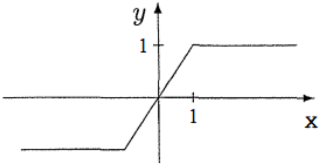
Fundamental Properties: Existence and Uniqueness

Example

Today, we will continue the Lipschitz continuity of the following function

$$f(x) = \begin{bmatrix} x_2 \\ -\mathrm{sat}(x_1 + x_2) \end{bmatrix}$$

where the saturation function is defined as



$$\mathrm{sat}(x) = \begin{cases} x, & \text{if } |x| \leq 1 \\ \mathrm{sgn}(x), & \text{if } |x| > 1. \end{cases}$$

- f is not continuously differentiable on $\mathbb{R}^2$.
- Note that, the saturation function satisfies

$$|\mathrm{sat}(\xi) - \mathrm{sat}(\eta)| \leq |\xi - \eta|$$

Fundamental Properties: Existence and Uniqueness

Example

$$f(x) = \begin{bmatrix} x_2 \\ -\text{sat}(x_1 + x_2) \end{bmatrix}$$

- Using $\|\cdot\|_2$ in $\mathbb{R}^2$, we have

$$\begin{aligned}
\|f(x) - f(y)\|_2^2 &= \left\| \begin{bmatrix} x_2 - y_2 \\ \text{sat}(y_1 + y_2) - \text{sat}(x_1 + x_2) \end{bmatrix} \right\|_2^2 \\
&\leq (x_2 - y_2)^2 + (\text{sat}(y_1 + y_2) - \text{sat}(x_1 + x_2))^2 \\
&\leq (x_2 - y_2)^2 + (y_1 + y_2 - x_1 - x_2)^2 \\
&\leq (x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2
\end{aligned}$$

Using the inequality[1]

$$a^2 + 2ab + 2b^2 = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \lambda_{\max}\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right) \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2^2$$

we conclude that

$$\|f(x) - f(y)\|_2 \leq \sqrt{2.618}\|x - y\|_2, \quad \forall x, y \in \mathbb{R}^2.$$

[1] For positive semi-define symmetric matrices, we have $x^\top Px \leq \lambda_{\max}(P)x^\top \cdot x, \ \forall x \in \mathbb{R}^n$.

Example

$$f(x) = \begin{bmatrix} x_2 \\ -\text{sat}(x_1 + x_2) \end{bmatrix}$$

- If we have used the more conservative inequality[2]

$$a^2 + 2ab + 2b^2 \le 2a^2 + 3b^2 \le 3(a^2 + b^2).$$

then, a more conservative (larger) Lipschitz constant ($L = \sqrt{3}$) will be obtained:

$$\|f(x) - f(y)\|_2 \le \sqrt{3}\|x - y\|_2, \quad \forall x, y \in \mathbb{R}^2.$$

[2] The following holds, for any $\lambda > 0$,

$$\left(\sqrt{\lambda}a - \frac{b}{\sqrt{\lambda}}\right)^2 = \lambda a^2 - 2ab + \frac{b^2}{\lambda} \ge 0 \iff 2ab \le \lambda a^2 + \frac{b^2}{\lambda}, \ \forall a, b \in \mathbb{R},$$

Fundamental Properties: Existence and Uniqueness

Example

Consider the scalar system

$$\dot{x} = -x^2, \text{ with } x(0) = -1$$

The function $f(x) = -x^2$ is locally Lipschitz for all $x \in \mathbb{R}$. Hence, it is Lipschitz on any compact subset of $\mathbb{R}$.

$$\frac{dx}{dt} = -x^2 \implies -\int_{\xi=-1}^{\xi=x} \xi^{-2} d\xi = \int_{\tau=0}^{\tau=t} d\tau \implies x(t) = \frac{1}{t-1}.$$

Therefore, the solution exists over [0,1]. As $t \to 1$, x(t) leaves any compact set. The phrase “finite escape time" is used to describe the phenomenon that a trajectory escapes to infinity at a finite time. In this example, we say that the trajectory has a finite escape time at $t = 1$.

Theorem (Global Existence and Uniqueness)

Suppose that $f(t,x)$ *is piecewise continuous in* t *and satisfies*

$$\|f(t,x) - f(t,y)\| \le L\|x - y\|, \quad \forall x, y \in \mathbb{R}^n, \ \forall t \in [t_0, t_1].$$

Then, the state equation $\dot{x} = f(t,x)$ *with* $x(t_0) = x_0$*, has a unique solution over* $[t_0, t_1]$*.*

Fundamental Properties: Existence and Uniqueness

Example

Conside the linear system

$$\dot{x} = A(t)x + g(t) = f(t,x)$$

where $A(t)$ and $g(t)$ are piecewise continuous functions of t.

Over a finite interval of time $[t_0, t_1]$, the elements of $A(t)$ are bounded. Hence, $\|A(t)\| \le a$, where $\|A\|$ is any induce matrix norm. The conditions of “Global Existence and Uniqueness Theorem" are satisfied since

$$\begin{aligned}\|f(t,x) - f(t,y)\| &= \|A(t)(x-y)\| \\ &\le \|A(t)\|\|x-y\| \le a\|x-y\|, \quad \forall x, y \in \mathbb{R}^n, \ \forall t \in [t_0, t_1].\end{aligned}$$

Therefore, this theorem guarantees that the system has a unique solution over $[t_0, t_1]$. Since t_1 can be arbitarily large, we can also conclude that the system has a unique solution $\forall t \ge t_0$. Hence, the system cannot have a finite escape time.

Fundamental Properties: Existence and Uniqueness

Example

Consider the scalar system

$$\dot{x} = -x^3 = f(x).$$

The function f(x) does not satisfy a global Lipschitz condition since the Jacobian $\left[\frac{\partial f}{\partial x}\right] = -3x^2$ is not globally bounded. Nevertheless, for any initial state $x(t_0) = x_0$, the equation has the unique solution and it can be obtained as

$$\frac{dx}{dt} = -x^3 \implies -\int_{\xi=x_0}^{\xi=x} \xi^{-3} d\xi = \int_{\tau=t_0}^{\tau=t} d\tau$$

$$\implies \left.\frac{\xi^{-2}}{2}\right|_{\xi=x_0}^{\xi=x} = \tau\Big|_{\tau=t_0}^{\tau=t}$$

$$\implies x(t) = \mathrm{sign}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}$$

Therefore, the equation $\dot{x} = -x^3$ has a unique solution over $t \geq t_0$ which means that the solution is well-defined over $t \geq t_0$.

Fundamental Properties: Existence and Uniqueness

From the last example, we see that "Global Existence and Uniqueness Theorem" is not a necessary and sufficient condition.

We have the following implications:

$\mathcal{C}^1$ class of continuously differentiable functions $\Longrightarrow$ locally Lipschitz

locally Lipschitz $\not\Longleftarrow$ $\mathcal{C}^0$ class of continuous functions (Ex: $f(x)=\sqrt[3]{x}$)

locally Lipschitz $\Longrightarrow$ $\mathcal{C}^0$ class of continuous functions

Second Order Time-Invariant Systems

In this section, we will present the methods for analyzing the behavior of second order time-invariant systems ($n = 2$):

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Phase plane is the plane with x_1 and x_2 along the coordinate axis as below.

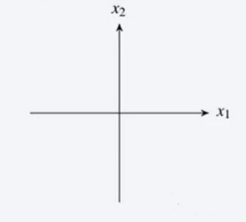


Figure: Phase Plane.

Second Order Time-Invariant Systems

Having an IVP, the solution can be represented by a curve in the phase plane which is called a trajectory from x_0 as shown the following figure.

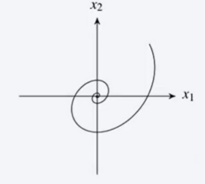


Figure: Trajectory.

If we draw the trajectories for several different IVs, we get a family of trajectories and this allows us to see how the system behaves for different IVs. This family of trajectories is called a phase portrait as shown below.

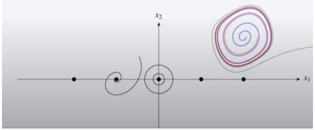


Figure: Phase Portrait.

Second Order Time-Invariant Systems

Analyzing the system behavior in this way by constructing a phase portrait is called a phase plane analysis.

The equilibrium points in two dimensional systems are denoted by **singular points.** These points are called singular because these are the only points in the **phase plane** where the slope of the trajectory is not well-defined:

$$x^* \in \mathbb{R}^2 \quad \text{such that} \quad \begin{bmatrix} f_1(x_1^*, x_2^*) \\ f_2(x_1^*, x_2^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The slope/tangent of a trajectory:

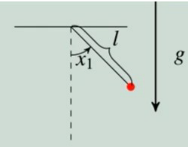
$$\left.\frac{dx_2}{dx_1}\right|_{(x_1,x_2)=(x_1^*,x_2^*)} = \left.\frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}}\right|_{(x_1,x_2)=(x_1^*,x_2^*)} = \left.\frac{f_2}{f_1}\right|_{(x_1,x_2)=(x_1^*,x_2^*)} \left(\frac{0}{0} \text{ undetermined}\right)$$

Second Order Time-Invariant Systems: Phase Portraits

There are two ways of performing a phase portrait analysis: analytically and/or computationally. In this lecture, we will learn how to perform a phase portrait analysis analytically. Let us consider the following example.

Example (Pendulum without Friction)

Let us consider the pendulum equation without friction:



$$m\ell\ddot{\theta} = -mg\sin\theta$$

A natural choice of the states will be $x_1 = \theta$, $x_2 = \dot{\theta}$. So, the state space model will be

$$\dot{x}_1 = x_2 = f_1(x_1, x_2) \implies x_2^* = 0,$$
$$\dot{x}_2 = \frac{g}{\ell}\sin x_1 = f_2(x_1, x_2) \implies x_1^* = k\pi, \ \forall k \in \mathbb{Z}.$$

and the equilibrium points will be $(x_1^*, x_2^*) = (k\pi, 0)$, $\forall k \in \mathbb{Z}$.

Second Order Time-Invariant Systems: Phase Portraits

The slope will be

$$\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = -\frac{g}{\ell}\frac{\sin\, x_1}{x_2} \implies \int x_2 dx_2 = -\frac{g}{\ell}\int \sin\, x_1 dx_1$$

$$\implies \frac{1}{2}x_2^2 = \frac{g}{\ell}\cos\, x_1 + c$$

Thus, we can draw the family of trajectories, i.e. the phrase portrait. In phrase portrait, you can not recover the solution x as a function of time from a trajectory. So, a phrase portrait does not give information about the quantitative (*Tür:* nicel) behavior of the system solutions, but gives information about the qualitative (*Tür:* nitel) behavior.

For those equilibrium points that are isolated, meaning that none of the neighboring points are equilibrium points, we do a local analysis. If the equilibrium point is part of a whole line of equilibrium points, for instance, then it is not an isolated one. We will see this further in detail in the next subsection.

Second Order Time-Invariant Systems: Local Analysis

Local analysis is equivalent to classify the equilibrium points in second order TI Systems. What we do then is to determine the qualitative behavior of the system near the equilibrium point,

- By first linearizing the system about the equilibrium point,
- Secondly finding the eigenvalues of the resulting system matrix A and "$\lambda(A)$",
- Finally classifying the equilibrium point.

Second Order Time-Invariant Systems: Local Analysis

1) Linearization about the equilibrium point: Given the system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

where $f_1, f_2 \in \mathcal{C}^1$ and an isolated equilibrium point $x^* = [x_1^*, x_2^*]^\top$, we have the following from Taylor's formula

$$f_i(x_1, x_2) = \underbrace{f_i(x_1^*, x_2^*)}_{=0} + \frac{\partial f_i}{\partial x_1}(x_1^*, x_2^*)\Delta x_1 + \frac{\partial f_i}{\partial x_2}(x_1^*, x_2^*)\Delta x_2 + \underbrace{P(\Delta x_1, \Delta x_2)}_{\text{Higher order terms}\approx 0}$$

for $i = 1, 2$ with $\Delta x = \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$. We can thus approximate the given second order system with the following linear system

$$\Delta \dot{x} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^*, x_2^*) & \frac{\partial f_1}{\partial x_2}(x_1^*, x_2^*) \\ \frac{\partial f_2}{\partial x_1}(x_1^*, x_2^*) & \frac{\partial f_2}{\partial x_2}(x_1^*, x_2^*) \end{bmatrix}}_{=:A} \Delta x$$

and A is called the Jacobian of f. Note that, this approximated system is a valid approximation when the system states are sufficiently close to the equilibrium point.

Second Order Time-Invariant Systems: Local Analysis

2) Find the Eigenvalues $\lambda(A)$: Solve $\det(\lambda I - A) = 0$. Since, $n = 2$, you will find two eigenvalues: λ_1, λ_2

3) Classify the Equilibrium Points: In this step, there are two cases to consider.

- Let us consider the first case that $\lambda_1, \lambda_2 \in \mathbb{R}$. Then, we have three subcases to consider:
  - Consider the subcase that $\lambda_2 < \lambda_1 < 0$. Then, the equilibrium point is a stable node, so that all the trajectories converge to zero without any oscillations. This kind of systems are also called overdamped second order TI systems.

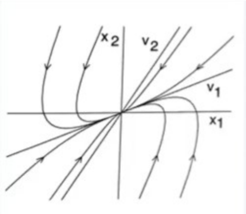


Figure: Stable node.

Second Order Time-Invariant Systems: Local Analysis

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3) Classify the Equilibrium Points: In this step, there are two cases to consider.

- Let us consider the first case that $\lambda_1, \lambda_2 \in \mathbb{R}$. Then, we have three subcases to consider:
  - Consider the subcase that $0 < \lambda_1 < \lambda_2$. Then, the equilibrium point is an unstable node and the phase portrait is demonstrated below.

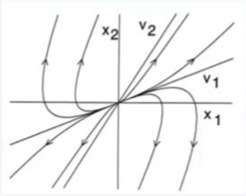

Figure: Unstable node.

Second Order Time-Invariant Systems: Local Analysis

2) Find the Eigenvalues $\lambda(A)$: Solve $\det(\lambda I - A) = 0$. Since, $n = 2$, you will find two eigenvalues: λ_1, λ_2

3) Classify the Equilibrium Points: In this step, there are two cases to consider.

- Let us consider the first case that $\lambda_1, \lambda_2 \in \mathbb{R}$. Then, we have three subcases to consider:
  - Consider the latter subcase that $\lambda_2 < 0 < \lambda_1$. Then, the equilibrium point is a saddle point and along v_1 which corresponds to the unstable eigenvalue λ_1, the trajectories move away the equilibrium point whereas along v_2, corresponding to the stable eigenvalue λ_2, the trajectories move towards the equilibrium point which is shown below.

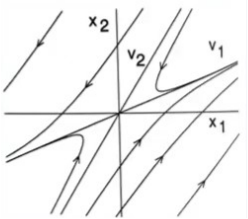

Figure: Saddle point.

Second Order Time-Invariant Systems: Local Analysis

3) Classify the Equilibrium Points: In this step, there are two cases to consider.

- Let us consider the second case that $\lambda_1, \lambda_2 \in \mathbb{C}$.
  - For this case, we will have complex conjugate eigenvalues, i.e. $\lambda_{1,2} = \alpha \pm j\beta$, $\alpha \in \mathbb{R}$, $\beta > 0$. For such kind of systems, the behavior of the system trajectories depend on the sign of the real part of the eigenvalues α:
    (a) For $\alpha < 0$, the equilibrium point of the system is a stable focus.
    (b) For $\alpha > 0$, the equilibrium point of the system is an unstable focus.
    (c) For $\alpha = 0$, the equilibrium point of the system is a center.

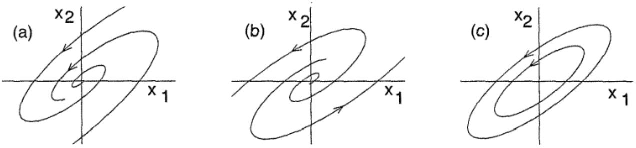

Figure: Phase portraits for (a) a stable focus; (b) an unstable focus; (c) a center.

Second Order Time-Invariant Systems: Local Analysis

Nonlinear second-order system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2), \quad f_1, f_2 \in C^1$$

Linear approximation of nonlinear 2^{nd} order system

$$\Delta\dot{x} = A \cdot \Delta x$$

Topological (*Tür:* Yapısal) Equivalence: If $Re(\lambda_{1,2}(A)) \neq 0$, then the local behavior of the nonlinear system is topologically equivalent with the behavior of the linear approximation of the nonlinear system. In other words, in a small neighborhood of the equilibrium point, we can approximate the behavior of the nonlinear system by the behavior of its linearization about this equilibrium point. Note that this is a **local** result.

Second Order Time-Invariant Systems: Local Analysis

Next week, we will perform a local analyis for the following example.

Example

The Jacobian matrix of the function

$$f(x) = \begin{bmatrix} \frac{1}{C}[-h(x_1) + x_2] \\ \frac{1}{L}[-x_1 - Rx_2 + u] \end{bmatrix}$$

with circuit parameters $u = 1.2$ V, $R = 1.5\ k\Omega = 1.5 \times 10^3\ \Omega$, $C = 2\ pF = 2 \times 10^{-12}\ F$ and $L = 5\ \mu H = 5 \times 10^{-6}\ H$ of the tunnel-diode circuit of Example 2.1 in [Khalil, 2002] is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0,5h'(x_1) & 0,5 \\ -0,2 & -0,3 \end{bmatrix}$$

where

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5,$$

$$h'(x_1) = \frac{dh}{dx_1} = 17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4.$$