MTM5135-Nonlinear Dynamical Systems

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Week 8

Lyapunov Functions and Lyapunov's Stability Theorem

Consider the TI system $\dot{x} = f(x)$ where $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, $x = 0 \in D$ is an equilibrium point of the system. To use Lyapunov's direct method, we need to find a function that can be used instead of energy to determine the stability properties of the equilibrium point. This function is called a Lyapunov function candidate.

Definition (Lyapunov Function)

$V : D \to \mathbb{R}$ is a **Lyapunov function** for $x = 0$ if and only if

- i) $V \in \mathcal{C}^1$
- ii) $V(0) = 0$ and $V(x) > 0$ in $D \setminus \{0\}$ (positive definiteness in D)
- iii) $\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) \leq 0$ in D (negative semi-definiteness in D)

If, moreover,

- iv) $\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) < 0$ in D $\setminus \{0\}$,

then, V is a **strict Lyapunov function** for $x = 0$.

Remark

If (i) and (ii) are satisfied, for a $V : D \to \mathbb{R}$, then V is called a ***Lyapunov function candidate****. In addition to (i) and (ii), when (iii) or (iv) are satisfied, the function V will no longer a "candidate" and named as Lyapunov function or strict Lyapunov function, respectively. Note also that, (iii) and (iv) are the only conditions where the system dynamics come into the criterion. The first two conditions only work out the function itself and the system dynamics do not come in here.*

Now, we are ready to present the sufficient conditions for stability and asymptotic stability of the equilibrium point.

Theorem (Lyapunov's Direct Method)

- *If there exists a* ***Lyapunov function*** *for $x = 0$, then $x = 0$ is* ***stable****.*
- *If there exists* ***strict Lyapunov function*** *for $x = 0$, then $x = 0$ is* ***asymptotically stable****.*

Proof.

Lyapunov Functions and Lyapunov's Stability Theorem

Question: How to apply Lyapunov's direct method for general $\dot{x} = f(x)$?

1) Choose a Lyapunov function "candidate" $V(x)$
   - For electrical/mechanical systems

$$V(x) = \text{Total Energy}$$

   - For others quadratic forms are generally used:

$$V(x) = \frac{1}{2}x^T P x = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^\top \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{12} & p_{22} & \dots & p_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ p_{1n} & p_{2n} & \dots & p_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

$$= \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + \dots$$

   - There are also some other methods for choosing $V(x)$

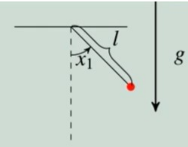
2) Determine whether $V(x)$ is a **Lyapunov function** or a **strict Lyapunov function** for the equilibrium point.
3) If the answer is "yes", then the equilibrium point is **stable** or **asymptotically stable**. If the answer is "no", then go back to step 1.

Remark

Failing to establish a Lyapunov function does not mean that the equilibrium point is unstable. The result of Lyapunov's direct method contains sufficent conditions not necessary conditions. There exists instability theorem for establishing an equilibrium point is unstable, e.g. Chetaev's theorem (see Theorem 4.3 in Khalil's book), but it will not be covered here.

Example (Pendulum without Friction)

Next week, we will consider the system governed by pendulum without friction:



$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1\end{aligned}$$

Spoiler Alert: Since this is a mechanical system it is natural to try using the energy as a Lyapunov function!