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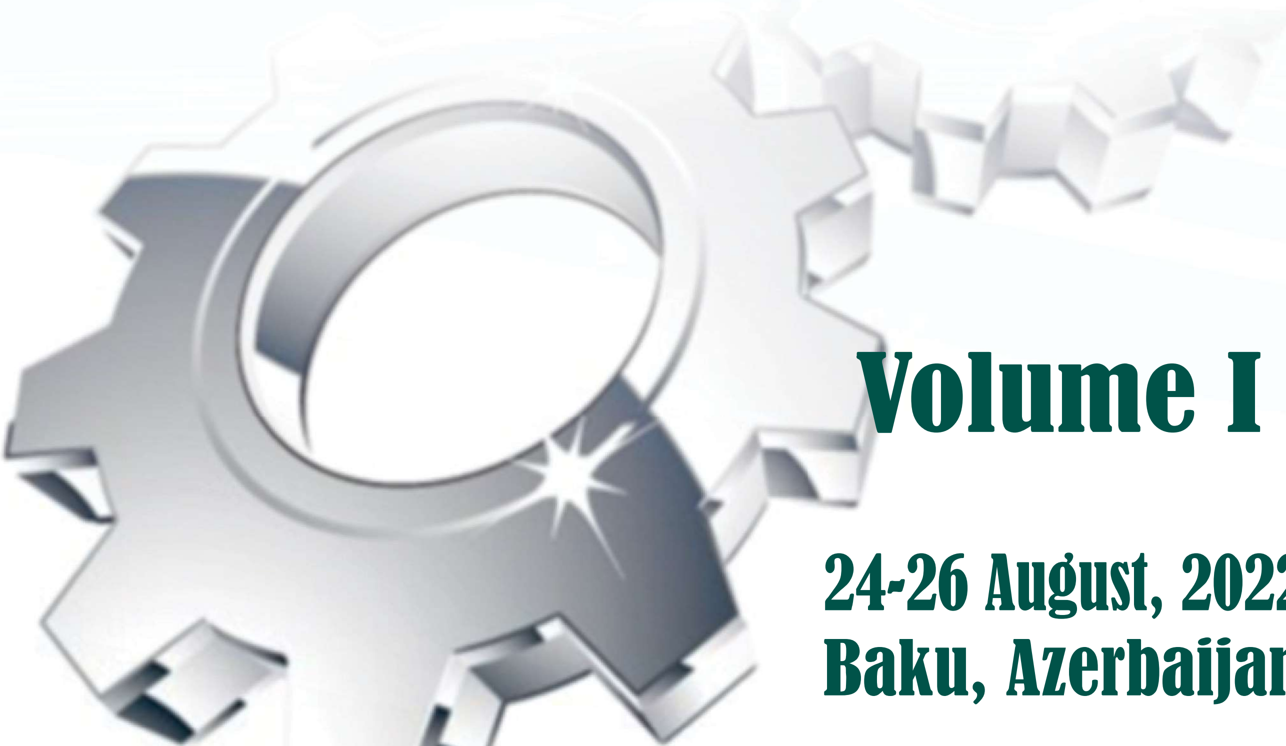

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TIME OPTIMAL CONTROL PROBLEM FOR SECOND ORDER LINEAR TIME INVARIANT SYSTEMS

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1. Introduction

The article discusses the time optimal control problem described by a second-order linear differential equation, which will undoubtedly play an important role in many technology advancements, for example, in aerospace, communications, automotive, and computer engineering, forced oscillations and resonance phenomenon, earthquake-induced vibrations of multi-storey buildings, etc. The significant advantage of modern control theory [1, 2, 4, 10] over the classical theory is its applicability to control problems involving time invariant and time-varying systems. Since the behaviour of many systems of importance in engineering practice is governed by second order controllable systems, our objective in the present paper is to contribute to the analysis of time optimal control problem of second order linear differential time invariant (LTI) systems. This is possible due to the Kalman type condition [2], the so-called "condition for generality of position" found for the second-order linear optimal control theory, which in a particular case for the first order linear theory coincides with the well-known Pontryagin's condition for generality of position [10]. To this end some qualitative results with respect to the solution of a given system is investigated and the Pontryagin's maximum principle for a problem with second order system of nonlinear differential equations are formulated. Then it is proved that the maximum principle uniquely determines the control function for each nontrivial solution of the second order adjoint equation; it turns out that optimal control is piecewise-constant, its only values are the vertices of the given polyhedron. To find the general solution of nonhomogeneous second order linear systems is used the method of variation of parameters. Then the existence and uniqueness theorems are formulated. At the end of the article, as an application of the main results, one well-known example is given. Finally, we conclude that the proposed method for studying the theory of optimal control can be the best alternative to previously existing methods with first-order equations.

2. Time optimal control for second order equations and the condition for generality of position

At the beginning we briefly reformulate the classical Pontryagin's maximum principle for first order differential equations to a problem with second order nonlinear differential equations:

$$\begin{gathered}\min \int_{t_0}^{t_1} f^0(x,x',u)dt, \quad \frac{d^2x}{dt^2} = f(x,x',u),\ u \in U \\ x(t_0)=x_0,\ x'(t_0)=x_1;\ x(t_1)=y_0,\ x'(t_1)=y_1,\end{gathered} \tag{1}$$

where $f(x, x', u)$ is the vector with coordinates $f^i(x, x', u)$ $(i = 1, ..., n)$, $x = (x^1, ..., x^n) \in \mathbb{R}^n$, $U \subset \mathbb{R}^r$ is the control domain. They are assumed to be continuous with respect to the set of variables $x^1, ..., x^n, x^{1'}, ..., x^{n'}, u$ and to be continuously differentiable with respect to $x^1, ..., x^n, x^{1'}, ..., x^{n'}$. The function $f^0(x, x', u)$ and its partial derivatives with respect to x^i and $x^{i'}$ $(i = 1, ..., n)$ are continuous. We shall regard as feasible any piecewise-continuous controls. We say that the feasible control function $u(t)$, $t_0 \leq t \leq t_1$ takes the point from the position (x_0, x_1) to the position (y_0, y_1) if the corresponding solution $x(t)$ satisfying the initial condition $x(t_0) = x_0$, $x'(t_0) = x_1$ is defined throughout the interval $t_0 \leq t \leq t_1$ and also satisfies the endpoint condition $x(t_1) = y_0$, $x'(t_1) = y_1$.

Now, we start with a rigorous statement of the problem. In what follows, we shall consider an entity, the law of motion of which is described by the following matrix equations:

$$x'' = A_0 x + A_1 x' + Bu, \quad u \in U, \tag{2}$$

where A_0, A_1 and B are matrices with dimensions $n \times n$, $n \times n$ and $n \times r$, respectively. $U \subset \mathbb{R}^r$ is a convex closed polyhedron. First, by change of variables the homogeneous second order linear system is reduced to the first order system and is investigated its eigen-analysis and structure of solutions set.

Theorem 1 (Characteristic equations). Let $x'' = A_0 x + A_1 x'$ be given with constant $n \times n$ matrices A_0, A_1 and $0_{n\times n}$, E_n be $n \times n$ zero and identity matrices, respectively. Besides, let $v' = Cv$ be its corresponding first order linear system, where $C = \begin{bmatrix} 0_{n\times n} & E_n \\ A_0 & A_1 \end{bmatrix}$, $v = \begin{bmatrix} x \\ x' \end{bmatrix}$ be $2n \times 2n$ block and $2n \times 1$ column matrices. Then $\det(C - \lambda E_{2n}) = (-1)^n \det\left[A_0 + \lambda(A_1 - \lambda E_n)\right]$.

Theorem 2. Let A_0, A_1 be $n \times n$ constant matrices and $C = \begin{bmatrix} 0_{n\times n} & E_n \\ A_0 & A_1 \end{bmatrix}$ be $2n \times 2n$ block matrices. Then $(C - \lambda E_{2n}) \begin{bmatrix} z & y \end{bmatrix}^\top = 0$ if and only if $(A_0 + \lambda A_1) z = \lambda_2 z$, $y = \lambda z$.

In this paper we consider only time optimal control problem for (2). Let us introduce matrices G_0, G_1, ..., G_{n-1} setting $G_0 = B$, $G_1 = A_1 B$, ..., $G_k = A_0 G_{k-2} + A_1 G_{k-1}$, ...; $k = 2, ..., n-1$. We say that the condition for generality of position is satisfied if, given any rib w of the polyhedron U, the vectors

$$G_0 w, \ G_1 w, \ ..., G_{n-1} w \tag{3}$$

are linearly independent in the space $\mathbb{R}^n$. Obviously, if $A_0 \equiv 0_{n\times n}$, then $G_k = A_1 G_{k-1}$ and $G_0 = B$, $G_1 = A_1 B$, $G_2 = A_1^2 B$, ..., $G_k = A_1^k B$, ...; $k = 0, ..., n-1$ i.e. vectors (3) are the same as in classical theory of time optimal control problem [10] for a first order linear systems with constant matrices: Bw, $A_1 Bw$, $A_1^2 Bw$, ..., $A_1^{n-1} Bw$. In what follows, we assume that the condition for generality of position is satisfied. The adjoint equation consists of the following

$$x^{*'} = A_0^* x^* - A_1^* x^{*'}. \tag{4}$$

Notice that the Euler-Lagrange type an adjoint equation (4) can be constructed by using the Mahmudov's adjoint inclusion [3-9]. Let us denote the maximum of $\langle x^*, Bu \rangle$ regarded as a function of the variable $u \in U$ by $P(x^*)$:

$$\langle x^*(t), Bu(t) \rangle = P(x^*(t)). \tag{5}$$

Theorem 3. For each nontrivial solution $x^*(t)$ of (4), the maximum principle defines uniquely a control function $u(t)$; moreover, the function $u(t)$ is piecewise-constant and its only values are the vertices of the polyhedron U.

Theorem 4. Suppose that the control domain U is the parallelepiped and that all the characteristic values of the pair of matrices (A_0, A_1), of equation (2) are real. Then, for each non-trivial solution $x^*(t)$ of second order adjoint equation (4), the relation (5) uniquely defines a control function $u(t) = (u^1(t), u^2(t), ..., u^r(t))$; moreover, each of the functions $u^k(t)$, $k = 1, ..., r$

is piecewise-constant and has not more than $n-1$ switching instants (i.e. not more than n interval of constancy), where n is the order of system (2).

Theorem 5. Suppose that $u_1(t)$, $t_0 \le t \le t_1$; $u_2(t)$, $t_0 \le t \le t_2$ are two optimal controls, taking the point (x_0, x_1) to the same point (y_0, y_1). Then the control $u_1(t)$ coincides with the control $u_2(t)$, i.e. $t_1 = t_2$ and $u_1(t) \equiv u_2(t)$ on the interval $t_0 \le t \le t_1$.

Theorem 6. Suppose that the coordinate origin of space $\mathbb{R}^r$ is an interior point of the polyhedron U, and that $u_1(t)$, $t_0 \le t \le t_1$; $u_2(t)$, $t_0 \le t \le t_2$ are two extremal controls, taking the point (x_0, x_1) to the coordinate origin $(y_0, y_1) = (0,0)$ of space $\mathbb{R}^{2n}$. Then the control $u_1(t)$ coincides with the control $u_2(t)$, i.e. $t_1 = t_2$ and $u_1(t) \equiv u_2(t)$ on the interval $t_0 \le t \le t_1$.

In this section by analogy with existence Theorem 13 [10] we can show that if there exists at least one feasible solution, then an optimal control function also exists.

Theorem 7. Assume that the condition for generality of position is satisfied. Then if there exists for the process described by equation (2) at least one feasible control function taking the point from the position (x_0, x_1) to the position (y_0, y_1), then an optimal control also exists.

Example 1. The following example studied without reducing it to a system of first-order differential equations.

$$\begin{aligned} &\inf\ t_1 = \int_{t_0}^{t_1} t \cdot dt \text{ subject to },\quad x'' = u,\ |u| \le 1, \\ &x(0) = x_0,\ x'(0) = x_1;\ x(t_1) = x'(t_1) = 0. \end{aligned} \tag{6}$$

The solution of this problem in the plane xt (not in the phase plane x^1x^2) consists of either one parabola or parts of two parabolas.

Keywords: Eigen Analysis, Existence, Generality Of Position, Maximum Principle, Second Order, Uniqueness.

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