Analysis of Integral Input-to-State Stable time-delay systems in cascade

G. Göksu[1,2] A. Chaillet[2,3]

[1]Department of Mathematics
Istanbul Technical University

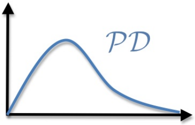
[2]Laboratoire des Signaux et Systèmes
CentraleSupélec, Univ. Paris-Saclay

[3]Institut Universitaire de France

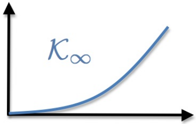
July 13-17, 2020 / 21st IFAC World Congress

Outline

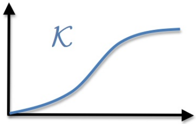
1. Preliminaries
   - Comparison functions
   - Notations
   - iISS for TDS
   - Sufficient condition for iISS
2. Cascade Interconnection
   - Problem Statement
   - Results in Delay-Free Context
   - Main result
   - GAS+iISS+Growth Rate Condition⇒GAS
3. Illustrative Example



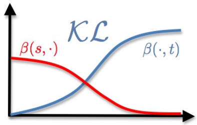
$$\begin{cases} \alpha \text{ continuous} \\ \alpha(0) = 0 \\ \alpha(s) > 0, \ \forall s > 0 \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{K} \\ \lim_{s \to \infty} \alpha(s) = \infty \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{PD} \\ \alpha \text{ increasing} \end{cases}$$

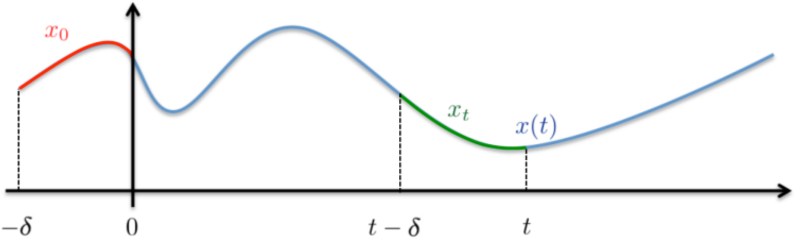


$$\begin{cases} \beta(\cdot, t) \in \mathcal{K}, \ \forall t \geq 0 \\ \beta(s, \cdot) \text{ nonincreasing}, \ \forall s \geq 0 \\ \lim_{t \to \infty} \beta(s, t) = 0, \ \forall s \geq 0 \end{cases}$$

Consider the nonlinear TDS: $\dot{x}(t) = f(x_t, u(t))$

- State History: $x_t \in \mathcal{C}^n$ defined with the maximum delay $\delta \geq 0$ as

$$x_t(s) := x(t+s), \quad \forall s \in [-\delta, 0].$$



- $\mathcal{C}$: Set of all continuous functions $\varphi : [-\delta; 0] \to \mathbb{R}$.
- $\mathcal{U}$: Set of measurable essentially bounded signals to $\mathbb{R}^m$.
- Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.
- Given any $\phi \in \mathcal{C}^n$, $\|\phi\| := \sup_{\tau \in [-\delta, 0]} |\phi(\tau)|$.
- $f : \mathcal{C}^n \times \mathbb{R}^m \to \mathbb{R}^n$, Lipschitz on bounded sets and to satisfy $f(0,0) = 0$.

- Lyapunov-Krasovskii functional (LKF) candidate: Any functional $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, Lipschitz on bounded sets, for which there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{C}^n. \tag{6}$$

- Its upper-right Dini derivative along the solutions of $\dot{x}(t) = f(x_t, u(t))$ is then defined for all $t \geq 0$ as

$$D^+ V(x_t, u(t)) := \limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}. \tag{7}$$

Definition (0-GAS)

The TDS is said to be globally asymptotically stable in the absence of inputs (0-GAS) if there exists $\beta \in \mathcal{KL}$ such that, the solution of the input-free system $\dot{x}(t) = f(x_t, 0)$ satisfies

$$|x(t)| \leq \beta(\|x_0\|, t), \qquad \forall t \geq 0.$$

Definition (iISS, (Pepe, Jiang, SCL, 2006))

The TDS is said to be integral input-to-state stable (iISS) if there exists $\beta \in \mathcal{KL}$ and $\nu, \sigma \in \mathcal{K}_\infty$ such that, its solution satisfies

$$|x(t)| \leq \beta(\|x_0\|, t) + \nu\left(\int_0^t \sigma(|u(s)|)ds\right), \quad \forall t \geq 0.$$

- Forward completeness (Hale, 1977, Theorem 3.2, p. 43)
- Asymptotic stability in the absence of inputs (0-GAS)

Definition (BEBS, BECS)

The TDS is said to have the bounded energy-bounded state (BEBS) property, if there exists $\zeta \in \mathcal{K}_\infty$ such that its solution satisfies

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \sup_{t \geq 0} |x(t)| < \infty.$$

It is said to have the bounded energy-converging state (BECS) property if there exists $\zeta \in \mathcal{K}_\infty$ such that, its solution satisfies

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \lim_{t \to \infty} |x(t)| = 0.$$

Proposition (iISS $\Rightarrow$ 0-GAS, BEBS, BECS)

If the TDS is iISS, then it is BEBS and BECS.

Proposition (iISS LKF, Necessity: (Lin, Wang, CDC, 2018), Sufficiency: (Pepe, Jiang, SCL, 2006))

The TDS is iISS if and only if there exists a LKF candidate $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{PD}$ *and* $\gamma \in \mathcal{K}_\infty$, *such that the following holds:*

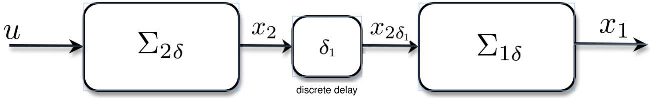
$$D^+ V(x_t, u(t)) \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \quad \forall t \geq 0.$$

$\rightarrow$ Finite-dimensional case: (Angeli et al., IEEE TAC, 2000).

Proposition (Sufficient Condition for iISS, (Chaillet, Pepe, CDC, 2018))

The TDS is iISS if there exists a LKF candidate $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{PD}$ *and* $\eta, \gamma \in \mathcal{K}_\infty$, *such that the following holds:*

$$D^+ V(x_t, u(t)) \leq -\frac{\alpha(|x(t)|)}{1 + \eta(\|x_t\|)} + \gamma(|u(t)|), \quad \forall t \geq 0.$$



Consider two nonlinear TDS in cascade:

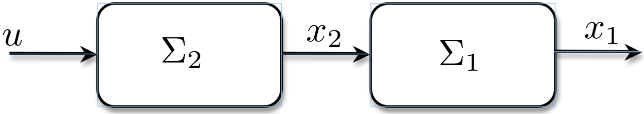
$$\Sigma_{1\delta} \; : \quad \dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1)), \tag{9a}$$

$$\Sigma_{2\delta} \; : \quad \dot{x}_2(t) = f_2(x_{2t}, u(t)), \tag{9b}$$

$\rightarrow \delta_1 \in [0, \delta]$: Interconnection through discrete delay.

Questions:

- iISS preserved under cascade interconnected TDS?
- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and BEBS?



Consider two nonlinear systems in cascade:

$$\begin{aligned}\Sigma_1 &: \quad \dot{x}_1 = f_1(x_1, x_2)\\ \Sigma_2 &: \quad \dot{x}_2 = f_2(x_2, u)\,.\end{aligned}$$

- ISS is naturally preserved in cascade [Sontag, EJC, 1995]
- iISS is not preserved by cascade [Panteley, Loría, Automatica, 2001], [Arcak et al., SIAM JCO, 2002].

Questions:

- ~~iISS preserved under cascade interconnected TDS?~~
- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and BEBS?

Theorem (Chaillet, Angeli, SCL, 2008)

Let V_1 and V_2 be two Lyapunov functional candidates. Assume that there exist $\gamma_1, \gamma_2 \in \mathcal{K}$, and $\alpha_1, \alpha_2 \in \mathcal{PD}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $u \in \mathbb{R}^m$,

$$\begin{aligned} \frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, x_2) &\leq -\alpha_1(|x_1|) + \gamma_1(|x_2|) \\ \frac{\partial V_2}{\partial x_2}(x_2) f_2(x_2, u) &\leq -\alpha_2(|x_2|) + \gamma_2(|u|) \,. \end{aligned}$$

$\rightarrow q_2(s) = \mathcal{O}_{s \to 0^+}(q_1(s))$: Given $q_1, q_2 \in \mathcal{PD}$, we say that q_1 *has greater growth* than q_2 around zero if $\exists\, k \geq 0$ such that $\limsup_{s \to 0^+} q_2(s)/q_1(s) \leq k$.

Questions:

- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and BEBS?

} Above condition valid for TDS?

Theorem

Assume that $\exists$ *two LKF candidates* $V_i : \mathcal{C}^{n_i} \to \mathbb{R}_{\geq 0}$ *and* $\eta_i \in \mathcal{K}_\infty$, $i \in \{1, 2\}$, *such that the following holds along any solution of* $\dot{x}_1(t) = f_1(x_{1t}, u_1(t))$

$$D^+ V_1(x_{1t}, u_1(t)) \leq -\frac{\alpha_1(|x_1(t)|)}{1 + \eta_1(V_1(x_{1t}))} + \gamma_1(|u_1(t)|) \tag{10}$$

and the following holds along any solution of $\dot{x}_2(t) = f_2(x_{2t}, u(t))$

$$D^+ V_2(x_{2t}, u(t)) \leq -\frac{\alpha_2(|x_2(t)|)}{1 + \eta_2(V_2(x_{2t}))} + \gamma_2(|u(t)|) \tag{11}$$

for all $t \geq 0$.

$$\gamma_1(s) = \mathcal{O}_{s \to 0^+}(\alpha_2(s)). \tag{12}$$

Then, the cascade is 0-GAS and satisfies the BEBS property.

Lemma

Let $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ be a LKF candidate satisfying, along any solution of the TDS $\dot{x}(t) = f(x_t)$,

$$D^+ V(x_t) \leq -\frac{\alpha(|x(t)|)}{1 + \eta(V(x_t))}, \tag{14}$$

for some $\alpha \in \mathcal{PD}$ and $\eta \in \mathcal{K}_\infty$. Let $\tilde{\alpha} \in \mathcal{PD}$ satisfying

$$\tilde{\alpha}(s) = \mathcal{O}_{s \to 0^+}(\alpha(s)). \tag{15}$$

Then, $\exists$ a continuously differentiable function $\rho \in \mathcal{K}_\infty$ such that the functional $\tilde{V} := \rho \circ V$ satisfies

$$D^+ \tilde{V}(x_t) \leq -\tilde{\alpha}(|x(t)|).$$

Proof of Lemma (Sketch).

- Take continuous non-decreasing function $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $q(s) > 0$ for all $s > 0$ such that ρ can be written as $\rho(s) = \int_0^s q(r)dr$ for all $s \geq 0$ and choose $\tilde{V} = \rho \circ V$.
- Its Dini derivative along the solutions of $\dot{x}(t) = f(x_t)$ reads
$$D^+\tilde{V}(x_t) \leq -q(V(x_t))\frac{\alpha(|x(t)|)}{1+\eta(V(x_t))}.$$
- Define $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as $\mu(s) := \sup_{r\in[0,s]} \frac{\tilde{\alpha}(r)}{\alpha(r)}, \quad \forall s \geq 0.$
- (15) ensures the boundedness of μ on $[0, a]$, $a > 0$.
- Choose $q(s) := \mu \circ \underline{\alpha}^{-1}(s)\big(1+\eta(s)\big), \ \forall s \geq 0.$
- Then, we have
$$\begin{aligned} q(V(x_t))\frac{\alpha(|x(t)|)}{1+\eta(V(x_t))} &\geq \mu \circ \underline{\alpha}^{-1}(V(x_t))\alpha(|x(t)|) \\ &\geq \mu(|x(t)|)\alpha(|x(t)|) \geq \tilde{\alpha}(|x(t)|). \end{aligned}$$

□

Proof of Theorem: Forward Completeness.

- (11) implies forward completeness of $\dot{x}_2(t) = f_2(x_{2t}, u(t))$.
- (10) with $u_1(t) = x_2(t - \delta_1) \Rightarrow \nexists$ any finite escape time for $x_1(t)$. □

Proof of Theorem: 0-GAS (Sketch).

- Consider the input-free system

$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t - \delta_1)), \tag{19a}$$

$$\dot{x}_2(t) = f_2(x_{2t}, 0). \tag{19b}$$

- (12)+Lemma $\Rightarrow \exists \rho \in \mathcal{K}_\infty \cap \mathcal{C}^1$ such that $\tilde{V}_2 := \rho \circ V_2$ satisfies

$$D^+ \tilde{V}_2(x_{2t}) \leq -2\gamma_1(|x_2(t)|). \tag{21}$$

- Now, consider the LKF defined as

$$\mathcal{V}_2(\phi_2) := \tilde{V}_2(\phi_2) + \int_{-\delta_1}^{0} \gamma_1(|\phi_2(\tau)|)d\tau, \quad \forall \phi_2 \in \mathcal{C}^{n_2}.$$

Proof of Theorem: 0-GAS (Sketch-Continued).

- In view of (21), its Dini derivative therefore reads

$$D^+\mathcal{V}_2(x_{2t}) \leq -\gamma_1(|x_2(t)|) - \gamma_1(|x_2(t-\delta_1)|). \tag{24}$$

- Furthermore (10) ensures that

$$D^+V_1(x_{1t}, x_2(t-\delta_1)) \leq -\frac{\alpha_1(|x_1(t)|)}{1+\eta_1(V_1(x_{1t}))} + \gamma_1(|x_2(t-\delta_1)|).$$

- Summing this with (24), we get that

$$D^+\mathcal{V}(x_t) \leq -\frac{\alpha_1(|x_1(t)|) + \gamma_1(|x_2(t)|)}{1+\eta_1(\mathcal{V}(x_t))},$$

□

Proof of Theorem: BEBS.

- (11) $\Rightarrow \exists \beta_2 \in \mathcal{KL}, \nu_2, \sigma_2 \in \mathcal{K}_\infty$ such that,
$$|x_2(t)| \leq \beta_2(\|x_{20}\|, t) + \nu_2\left(\int_0^t \sigma_2(|u(s)|)ds\right), \quad \forall t \geq 0. \tag{28}$$
- Assume that the following bounded energy holds for some $c \geq 0$.
$$\int_0^\infty \max\{\gamma_2(|u(\tau)|), \sigma_2(|u(\tau)|)\}d\tau \leq c \tag{29}$$
- Then, we ensure that $\lim_{t\to\infty} |x_2(t)| = 0$ and $\exists T := T_{x_{20},u} \geq 0$ such that $\|x_{2t}\| \leq 1, \ \forall t \geq T$, which guarantees that $V_2(x_{2t}) \leq \overline{\alpha}_2(1), \ \forall t \geq T$.
- Integrating the dissipation inequality (11) of V_2, we have, for all $t \geq T$,
$$\begin{aligned} V_2(x_{2t}) - V_2(x_{20}) &\leq -\int_0^t \frac{\alpha_2(|x_2(\tau)|)}{1+\eta_2(V_2(x_{2\tau}))}d\tau + \int_0^t \gamma_2(|u(\tau)|)d\tau \\ &\leq -\int_T^\infty \frac{\alpha_2(|x_2(\tau)|)}{\bar{\eta}_2}d\tau + \int_0^\infty \gamma_2(|u(\tau)|)d\tau, \end{aligned}$$

  where $\bar{\eta}_2 := 1 + \eta_2 \circ \overline{\alpha}_2(1)$.

Proof of Theorem: BEBS (Continued).

- From (29), $\int_T^\infty \alpha_2(|x_2(\tau)|)d\tau \leq (\overline{\alpha}_2(\|x_{20}\|) + c)\,\bar{\eta}_2$.
- From growth rate condition (12), $\exists k > 0$ s.t. $\gamma_1(s) \leq k\alpha_2(s)\ \forall s \in [0, 1]$.
- It follows that

$$\int_{-\delta_1}^{\infty} \gamma_1(|x_2(\tau)|)d\tau \leq \int_{-\delta_1}^{T} \gamma_1(|x_2(\tau)|)d\tau + \int_{T}^{\infty} k\alpha_2(|x_2(\tau)|)d\tau.$$

- Integrating dissipation inequality (10) with $u_1(t) = x_2(t - \delta_1)$, we have

$$\begin{aligned}\underline{\alpha}_1(|x_1(t)|) &\leq \overline{\alpha}_1(\|x_{10}\|) + \int_0^t \gamma_1(|x_2(\tau - \delta_1)|)d\tau \\ &\leq \overline{\alpha}_1(\|x_{10}\|) + \int_{-\delta_1}^{t-\delta_1} \gamma_1(|x_2(\tau)|)d\tau.\end{aligned}$$

- It holds that

$$\underline{\alpha}_1(|x_1(t)|) \leq \overline{\alpha}_1(\|x_{10}\|) + \int_0^T \gamma_1(|x_2(\tau)|)d\tau + \tilde{c}(\|x_0\|).$$

- The cascade owns the BEBS property. □

- The growth rate condition $\gamma_1(s) = \mathcal{O}_{s\to 0^+}(\alpha_2(s))$ is reminiscent of the one obtained in [Chaillet, Angeli, SCL, 2008].
- In [Chaillet, Angeli, SCL, 2008], it was shown that the growth rate condition implies iISS in finite-dimensional systems.
  - This is due to the fact that, 0-GAS+(a relaxed version of) BEBS implies iISS in finite-dimensional systems as presented in [Angeli et al., SIAM JCO, 2004].
  - Not yet been extended to TDS.
- The small-gain results for interconnected iISS TDS in [Ito et. al., Automatica, 2010]
  - involves the upper and lower bounds on V_1 and V_2, thus leading to a more conservative condition,
  - imposes that the dissipation rates for the driving and driven subsystems are of class class $\mathcal{K}$ (rather than $\mathcal{PD}$), meaning that both subsystems are required to have an ISS-like behavior for small inputs and
  - cannot be used for our illustrative example.

Consider the following input-free cascade:

$$\dot{x}_1(t) = f_1(x_{1t}, x_2(t-\delta_1)) \tag{13a}$$
$$\dot{x}_2(t) = f_2(x_{2t}). \tag{13b}$$

Corollary

Assume that there exist two LKF candidates $V_1 : \mathcal{C}^{n_1} \to \mathbb{R}_{\geq 0}$ *and* $V_2 : \mathcal{C}^{n_2} \to \mathbb{R}_{\geq 0}$, $\underline{\alpha}_i, \overline{\alpha}_i, \eta_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}$, $i \in \{1,2\}$, *and* $\gamma_1 \in \mathcal{K}_\infty$ *such that, the following holds along any solution of* $\dot{x}_1(t) = f_1(x_{1t}, u_1(t))$

$$D^+V_1(x_{1t}, u_1(t)) \leq -\frac{\alpha_1(|x_1(t)|)}{1+\eta_1(V_1(x_{1t}))} + \gamma_1(|u_1(t)|),$$

and the following holds along any solution of (13b)

$$D^+V_2(x_{2t}) \leq -\frac{\alpha_2(|x_2(t)|)}{1+\eta_1(V_2(x_{2t}))}, \qquad \forall t \geq 0.$$

Assume also that $\gamma_1(s) = \mathcal{O}_{s\to 0^+}(\alpha_2(s))$.

Example

Consider the following cascade TDS:

$$\dot{x}_1(t) = -\,\mathrm{sat}(x_1(t)) + \frac{1}{4}\mathrm{sat}(x_1(t-1)) + x_1(t)x_2(t-2)^2 \tag{34a}$$

$$\dot{x}_2(t) = -\,\frac{3}{2}x_2(t) + x_2(t-1) + u(t)\int_{t-1}^{t} x_2(\tau)d\tau. \tag{34b}$$

- $\mathrm{sat}(s) := \mathrm{sign}(s)\min\{|s|, 1\}$ for all $s \in \mathbb{R}$.
- $n_1 = n_2 = 1$, $m = 1$, $\delta_1 = \delta = 2$.

Consider the LKF candidates defined as

$$V_1(\phi_1) := \ln\left(1+\phi_1(0)^2 + \frac{1}{2}\int_{-1}^{0}\phi_1(\tau)\mathrm{sat}(\phi_1(\tau))d\tau\right), \tag{35a}$$

$$V_2(\phi_2) := \ln\left(1 + \phi_2(0)^2 + \int_{-1}^{0} \phi_2(\tau)^2 d\tau\right), \tag{35b}$$

By deriving, we have

$$D^+ V_1(x_{1t}, x_{2t}) \leq -\frac{x_1(t)\mathrm{sat}(x_1(t))}{1+\eta_1(V_1(x_{1t}))} + 2x_2(t-2)^2,$$

$$D^+ V_2(x_{2t}, u(t)) \leq -\frac{x_2(t)^2}{1+\eta_2(V_2(x_{2t}))} + |u(t)|.$$

where $\eta_1(s) = \eta_2(s) = e^s - 1$.The functions are

- $\alpha_1(s) = \mathrm{sat}(s)s$,
- $\alpha_2(s) = s^2$,
- $\eta_1(s) = \eta_2(s) = e^s - 1$,
- $\gamma_1(s) = 2s^2$ and
- $\gamma_2(s) = s$.

$\rightarrow$ Growth-rate condition: $2s^2 = \mathcal{O}_{s\rightarrow 0^+}(s^2)$.

The assumptions of Theorem are fulfilled.Thus, the cascade (35) is 0-GAS and owns the BEBS property.

- Conditions under which the cascade of two iISS TDS is 0-GAS and has the BEBS property.
- Growth restrictions on the input rate of the driven subsystem and the dissipation rate of the driving one.
- An academic example illustrates the applicability of the result.
- Limitations:
    - More generic interconnection of the form $\dot{x}_1(t) = f_1(x_{1t}, x_{2t})$.
    - Concluding that the overall cascade is iISS.
        - 0-GAS+BEBS$\Rightarrow$iISS for TDS?
    - Allowing the input u to impact directly the driven subsystem.

Acknowledgement

İTÜ

İSTANBUL TEKNİK ÜNİVERSİTESİ 1773

L2S
Laboratoire des Signaux & Systèmes

cnrs
CentraleSupélec
UNIVERSITÉ PARIS SUD
université PARIS-SACLAY

IUF
institut universitaire de France

TÜBİTAK

Contact

Gökhan Göksu	
E-Mail (ITU):	goksug@itu.edu.tr
E-Mail (L2S):	gokhan.goksu@centralesupelec.fr
Web:	web.itu.edu.tr/goksug

Antoine Chaillet	
E-Mail:	antoine.chaillet@centralesupelec.fr
Web:	sites.google.com/site/antoinechaillet/